

Introduction

In this note, we prove that $M := \mathrm{SL}(2, \mathbb{R}) / \mathrm{SL}(2, \mathbb{Z})$ is homeomorphic to the trefoil knot complement. We do that by showing they have the same Seifert-fibred structure: base orbifold an open disk with two conic points of orders 2 and 3. All of this is based on [RG79].

Fibering the Quotient

We will often work with $\mathrm{PSL}(2, \mathbb{R}) / \mathrm{PSL}(2, \mathbb{Z})$, which is the same space as M since

$$\mathrm{PSL}(2, \mathbb{R}) / \mathrm{PSL}(2, \mathbb{Z}) \cong (\mathrm{SL}(2, \mathbb{R}) / \{\pm I\}) / (\mathrm{SL}(2, \mathbb{Z}) / \{\pm I\}) \cong \mathrm{SL}(2, \mathbb{R}) / \mathrm{SL}(2, \mathbb{Z})$$

The reason to work with $\mathrm{PSL}(2, \mathbb{R}) / \mathrm{PSL}(2, \mathbb{Z})$ is only because most fibres in the Seifert fibering will have trivial stabilizer. Recall that a Seifert fibering comes from an action of S^1 on a space, such that each map $S^1 \rightarrow \mathcal{O}$ for any orbit $\mathcal{O}$ is a covering map with finite fiber. Regarding M as the set of left cosets of $\mathrm{PSL}(2, \mathbb{Z})$, we can define a left action of $S^1 \cong \mathrm{PSO}(2, \mathbb{R})$ simply via left multiplication. Note that if we let $\mathrm{SO}(2, \mathbb{R})$ act on M , then $-I$ would act trivially, and so every fiber would be exceptional.

We now find all the exceptional orbits of this action. Throughout the rest of this section, let $P \in \mathrm{PSO}(2, \mathbb{R})$, $A \in \mathrm{PSL}(2, \mathbb{R})$, and $S \in \mathrm{PSL}(2, \mathbb{Z})$ for yet-to-be-determined matrices. We will use $\hat{P}$ to denote a preimage in $\mathrm{SO}(2, \mathbb{R})$, and similarly for the other matrices. An exceptional orbit is simply one that has non-trivial stabilizer, which is equivalent to

$$PA = AS$$

Said another way, $A^{-1}PA = S$; that is, P is conjugate to an integer matrix. Working in $\mathrm{SO}(2, \mathbb{R})$, we have that

$$\hat{P} := T(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

A necessary condition for $\hat{P}$ to be conjugate to an integer matrix is that its trace is an integer, which implies $2 \cos \theta \in \{-2, -1, 0, 1, 2\}$. This implies that $\theta \in \{0, \pm \frac{\pi}{3}, \pm \frac{\pi}{2}, \pm \frac{2\pi}{3}, \pi\}$. However, we have that

$$\begin{aligned} T(0) &= I & T(-\frac{\pi}{3}) &= T(\frac{\pi}{3})^{-1} \\ T(\pi) &= -I & T(\frac{2\pi}{3}) &= -IT(\frac{\pi}{3})^{-1} \\ T(-\frac{\pi}{2}) &= -IT(\frac{\pi}{2}) & T(-\frac{2\pi}{3}) &= -IT(\frac{\pi}{3}) \end{aligned}$$

So, when considering exceptional orbits, we can focus attention on $T(\frac{\pi}{2})$ and $T(\frac{\pi}{3})$. We will write C and H for the images of $T(\frac{\pi}{2})$ and $T(\frac{\pi}{3})$ in $\mathrm{PSO}(2, \mathbb{R})$. Note that in $\mathrm{PSO}(2, \mathbb{R})$ these elements have order 2 and 3 respectively, so they do coincide with coverings with finite fiber. We will show below that each has exactly one exceptional orbit, so that the $\mathrm{PSO}(2, \mathbb{R})$ action does indeed give a Seifert fibering with two exceptional fibers.

Exceptional Fibers

Our main tool in analyzing the exceptional fibers will be the QR -decomposition of an element in $SL(2, \mathbb{R})$. In particular, we can write $\hat{A} = QR$ for $Q \in SO(2, \mathbb{R})$ and R an upper triangular matrix with positive diagonal. Since $\det(\hat{A}) = 1$, the diagonal entries of R are multiplicative inverses. For the orbit of the coset $A \backslash PSL(2, \mathbb{Z})$ to have C in its stabilizer, we need (as above) for $A^{-1}CA$ to be an integer matrix. If we write $\hat{A} = QR$, then we're asking to solve

$$R^{-1}(Q^{-1}\hat{C}Q)R = \hat{S}$$

Now $Q^{-1}\hat{C}Q \in SO(2, \mathbb{R})$, which is Abelian, so this is just $\hat{C}$. Similarly for H , we only need to solve $R^{-1}\hat{H}R = S$.

Let's do the $\hat{C}$ case first, since it's a little easier. We have

$$R^{-1}\hat{C}R = \begin{pmatrix} a^{-1} & -b \\ 0 & a \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} -ab & -(a^{-2} + b^2) \\ a^2 & ab \end{pmatrix}$$

For this to be an integer matrix, we first note that we can write $a = m^{1/2}$, for $m \in \mathbb{Z}^+$ (recall that $a > 0$). Then since $ab \in \mathbb{Z}$, we have $b = nm^{-1/2}$. Finally, we have

$$a^{-2} + b^2 = \frac{1 + n^2}{m} \in \mathbb{Z}$$

so that $n^2 \equiv -1 \pmod{m}$. It might seem there are potentially infinitely many different orbits stabilized by C , namely all those with different values of m and n . We will instead show all of these live in the same $SO(2, \mathbb{R})$ orbit as the case $m = n = 1$. Let

$$R_{1,1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

which is the $m = n = 1$ case. To show all others are in the same orbit, we need to show that for every m, n satisfying the conditions above we have $T(\theta) \in SO(2, \mathbb{R})$ and $S \in SL(2, \mathbb{Z})$ such that

$$T(\theta)R_{m,n} = R_{1,1}S$$

This is equivalent to $R_{1,1}^{-1}T(\theta)R_{m,n} \in SL(2, \mathbb{Z})$, so we compute

$$\begin{aligned} R_{1,1}^{-1}T(\theta)R_{m,n} &= \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \sqrt{m} & \frac{n}{\sqrt{m}} \\ 0 & \frac{1}{\sqrt{m}} \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{m}(\cos \theta - \sin \theta) & \frac{(n-1)\cos \theta - (n+1)\sin \theta}{\sqrt{m}} \\ \sqrt{m}\sin \theta & \frac{\cos \theta + n\sin \theta}{\sqrt{m}} \end{pmatrix} \end{aligned}$$

For this giant matrix to be in $SL(2, \mathbb{Z})$, the lower left corner and the identity $\cos^2 + \sin^2 = 1$ imply

$$\sin \theta = \frac{k}{\sqrt{m}}$$

$$\cos \theta = \pm \frac{\sqrt{m - k^2}}{\sqrt{m}}$$

Now since $n^2 + 1 \equiv 0 \pmod{m}$, we know (by working in $\mathbb{Z}[i]$) that we can write $m = k^2 + l^2$ for two integers k and l . We thus let $\cos \theta = \frac{l}{\sqrt{m}}$, where the sign of l will be chosen later. This already ensures the first column of our big matrix is integral. For the lower right corner to be an integer, we need $l + nk \equiv 0 \pmod{m}$. We already know that

$$l^2 \equiv -k^2 \equiv n^2 k^2 \pmod{m}$$

so $l \equiv \pm nk$ already. Now we choose the sign of l so that $l + nk$ is divisible by m . Finally, the upper right corner can be rewritten as

$$\frac{(n-1)l - (n+1)k}{m} = \frac{-(nk + l) + (nl - k)}{m}$$

We already know $nk + l$ is divisible by m ; since $n(nk + l) \equiv nl - k \pmod{m}$, our numerator is divisible by m , so our entire matrix is integral. We have thus shown that the only orbit stabilized by A is the one containing the $R_{1,1}$ $PSL(2, \mathbb{Z})$ coset.

Similar reasoning applies in the H case. We find that

$$R^{-1} \hat{H} R = \frac{1}{2} \begin{pmatrix} 1 - ab\sqrt{3} & \sqrt{3}(a^{-2} + b^2) \\ a^2\sqrt{3} & 1 + ab\sqrt{3} \end{pmatrix}$$

$$a = \frac{\sqrt{2m}}{\sqrt[4]{3}}$$

$$b = \frac{n}{\sqrt{2m}\sqrt[4]{3}}$$

$$n^2 \equiv -3 \pmod{4m}$$

We can similarly show that all $R_{m,n}$ are in the same orbit as $R_{1,1}$. We compute

$$R_{1,1}^{-1} T(\theta) R_{m,n} = \begin{pmatrix} \sqrt{m} \cos \theta - \frac{\sqrt{m}}{\sqrt{3}} \sin \theta & \frac{(3n-3) \cos \theta - (n\sqrt{3}+3\sqrt{3}) \sin \theta}{6\sqrt{m}} \\ 2\frac{\sqrt{m}}{\sqrt{3}} \sin \theta & \frac{3 \cos \theta + n\sqrt{3} \sin \theta}{3\sqrt{m}} \end{pmatrix}$$

For this to be integral, we have

$$\sin \theta = \frac{k\sqrt{3}}{2\sqrt{m}}$$

$$\cos \theta = \pm \frac{\sqrt{4m - 3k^2}}{2\sqrt{m}}$$

Just as before, if we work in $\mathbb{Z}[\omega]$ (where $\omega^3 = 1$), we show that the fact $n^2 + 3 \equiv 0 \pmod{4m}$ implies $4m = l^2 + 3k^2$ for some integers k and l , so that $\cos \theta = \pm \frac{l}{2\sqrt{m}}$. We will pick the sign of l later.

For the second column to be an integer, we have

$$\frac{3l + 3kn}{6m} \in \mathbb{Z}$$

$$\frac{3(nl - 3k) - 3(l + kn)}{12m} \in \mathbb{Z}$$

But we have

$$l^2 \equiv -3k^2 \equiv n^2 k^2 \pmod{4m}$$

so that $l \equiv \pm nk$, and we choose the sign of l so that $l + nk$ is divisible by $4m$, and thus $3(l + nk)$ is divisible by $12m$. That allows our second column to be integral as well. Thus the only orbit stabilized by H is the one containing the $R_{1,1} \text{PSL}(2, \mathbb{Z})$ coset.

The appearance of $\mathbb{Z}[i]$ and $\mathbb{Z}[\omega]$ is not coincidence. In fact, M is the moduli space of unimodular lattices in $\mathbb{R}^2$, and the exceptional orbits here are those coming from square and hexagonal lattices. We'll say more about this in a later section.

Base Orbifold

We have shown that M is a Seifert fibred space, with two exceptional fibers. We now determine the base orbifold. Let S be the base surface. The quotient $\text{PSL}(2, \mathbb{R}) / \text{PSL}(2, \mathbb{Z})$ gives an exact sequence

$$1 \rightarrow \pi_1(\text{PSL}(2, \mathbb{R})) \rightarrow \pi_1(M) \rightarrow \text{PSL}(2, \mathbb{Z}) \rightarrow 1$$

$\text{PSL}(2, \mathbb{R})$ deformation retracts to $\text{PSO}(2, \mathbb{R}) \cong S^1$, the group giving the fibers of M ; thus the above shows that $\pi_1^O(S) \cong \text{PSL}(2, \mathbb{Z})$, where π_1^O is the orbifold fundamental group. There is an epimorphism $\pi_1^O(S) \rightarrow \pi_1(S)$, and thus $\text{PSL}(2, \mathbb{Z})$ surjects onto $\pi_1(S)$, which implies the Abelianization surject. But by Bass-Serre theory $\text{PSL}(2, \mathbb{Z}) \cong C_2 * C_3$, which has Abelianization of order 6, so $H_1(S)$ has order at most 6 as well. Since S is boundaryless (as M is), there are only three possibilities: S^2 , $\mathbb{RP}^2$, and the open disk $\mathbb{D}^2$. We can rule the first two out via the known presentation of closed Seifert fibred fundamental groups, as follows:

- If $S \cong S^2$, then we would have

$$\pi_1(M) \cong \langle x, y, z \mid [x, z] = [y, z] = 1, x^2 = y^3 = z, xy = z^b \rangle$$

where $b \in \mathbb{Z}$. This is easily seen to be a finite cyclic group (generated by x), but $\pi_1(M)$ surjects onto $\text{PSL}(2, \mathbb{Z})$, so is infinite.

- If $S \cong \mathbb{RP}^2$, then we would have

$$\pi_1(M) \cong \langle a, x, y, z \mid z^a = z^{-1}, [x, z] = [y, z] = 1, x^2 = y^3 = z, xy a^2 = z^b \rangle$$

where $b \in \{0, 1\}$ since $\mathbb{RP}^2$ is not orientable, but M is. This gives the following presentation for the orbifold fundamental group (quotient out by $\langle z \rangle$):

$$\pi_1^O(S) \cong \langle a, x, y \mid x^2 = y^3 = 1, xy = a^{-2} \rangle$$

It's easy to check this group has Abelianization of order 12, and thus cannot be isomorphic to $\mathrm{PSL}(2, \mathbb{Z})$.

We thus see that $S \cong \mathbb{D}^2$, and the base orbifold of M is a disk with two conical points, one of order 2 and one of order 3.

Trefoil Complement

Now we show that the trefoil knot complement $N = S^3 \setminus K$ is also a Seifert-fibred space with the same base orbifold as above (a disk with two cone points of orders 2 and 3), so that $M \cong N$ (even via fiber-preserving diffeomorphism). Here K is the trefoil knot.

We can think of S^3 as the gluing of two solid tori, $S^3 = D_1 \cup_H D_2$, where $H : \partial D_1 \rightarrow \partial D_2$ reverses longitude and latitude (that is, it acts as $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$). If we think of K as living in ∂D_1 as a $(2, 3)$ torus knot, we can "refiber" D_1 so that K is the longitude. This fibering gives a fibering (via H) to D_2 as well, and thus we are left with a Seifert-fibred space. The central fiber of D_1 becomes an exceptional fiber of order 3, and the central fiber of D_2 becomes an exceptional fiber of order 2. The base orbifold is two disks (one for each D_i) glued along their boundaries; that is, a sphere. When we remove K , we remove a point from this sphere, so the base orbifold of N is a disk with two cone points of orders 2 and 3, as desired.

Another Perspective

This final section is meant to give a different approach to showing $\mathrm{SL}(2, \mathbb{R}) / \mathrm{SL}(2, \mathbb{Z})$ is a Seifert-fibred space, with base orbifold a topological disk with two cone points.

A lattice of $\mathbb{R}^2$ consists of all integral combinations of two linearly independent vectors: $\{mu + nv : m, n \in \mathbb{Z}\}$. Such a lattice is called unimodular if the parallelogram formed by u and v has area 1, which is to say $\det(u \ v) = 1$. Now $\mathrm{SL}(2, \mathbb{R})$ acts on the set of unimodular lattices, and in fact acts transitively: it moves any matrix of determinant 1 to any other

such matrix. The point stabilizer of the square lattice $\{(m, n) : m, n \in \mathbb{Z}\}$ is $\text{SL}(2, \mathbb{Z})$. Said another way, $\text{SL}(2, \mathbb{Z})$ acts trivially on all lattices, since it just finds new "bases" for the lattice. Thus $M = \text{SL}(2, \mathbb{R}) / \text{SL}(2, \mathbb{Z})$ is a topological representation of all unimodular lattices.

As above, we have $\text{SO}(2, \mathbb{R}) \subset \text{SL}(2, \mathbb{R})$ acting via rotations on these lattices, and this gives us a Seifert fibering. We'd like to identify the topological type of the base orbifold, and any exceptional fibers. The orbifold can be thought of as a set of double coset representatives $\text{SO}(2, \mathbb{R})x\text{SL}(2, \mathbb{Z})$. We find a set of representatives via the following construction. For a lattice L , first identify (one of) the closest vectors in L to the origin, and call this u . Now use $\text{SO}(2, \mathbb{R})$ to rotate this u to lie on the positive x -axis: $u = (r, 0)$. We now let v be (one of) the closest vectors to the origin not lying on the x -axis, say $v = (x, y)$. What are all the possible v ? Well, we know it cannot be closer to the origin than u , so it lives at least a distance r from the origin. We also know it has $|x| \leq \frac{r}{2}$, since otherwise we could shift v to $v \pm u$ to be an even closer vector. Since we can (after possibly replacing v with $-v$) assume $y > 0$, this gives us a region of $\mathbb{R}^2$ like below.

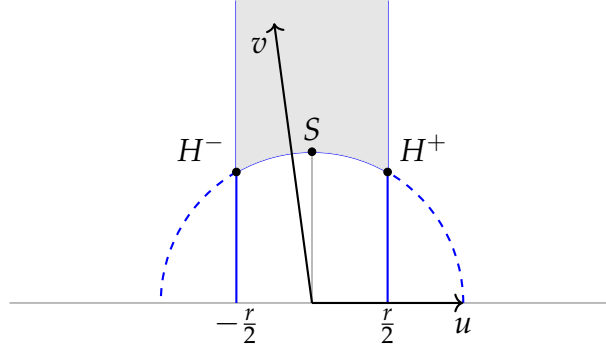


Figure 1: The fundamental region for unimodular lattices

For a given value of $r > 0$, only a single horizontal line of this region will give us a unimodular lattice (when $y = \frac{1}{r}$). However, as r varies, all the horizontal lines of this region give us all the double coset representatives we want. For example, the horizontal line through S gives all the possible v when $u = (1, 0)$, the point S corresponding to the square lattice. The horizontal line through H^- and H^+ gives only two possible v when $u = (\frac{\sqrt{2}}{\sqrt[4]{3}}, 0)$, and these give the hexagonal lattice. Note that both these points give us the same lattice, as the v for H^- corresponds to $v - u$ for H^+ . In fact, every point on the vertical line $x = -\frac{r}{2}$ should be identified with the corresponding point on $x = \frac{r}{2}$, since they differ only by adding u . Similarly, every point on the arc from H^- to S should be identified with its mirror image on the arc from S to H^+ : since both u and v are equidistant from the origin, our rotation via $\text{SO}(2, \mathbb{R})$ could instead have aligned v with the x -axis, thus (after negating u) changing the angle between the two vectors from θ to $\pi - \theta$.

It is a fact that this shaded region, after making the identifications above, gives a double-coset transversal. This follows from a Euclidean-like algorithm for finding the shortest

vector in a lattice, and only works for two-dimensional lattices. The shaded region, when we identify opposite vertical sides, and mirror images in the circular side, gives a punctured sphere, otherwise known as an open disk. So we have recovered the topological type of the base orbifold of M .

What about the exceptional fibers? For those we need to find lattices that have rotational symmetry. Surely those corresponding to S and H (the square and hexagonal lattices) exhibit a 4-fold and 6-fold rotational symmetry (which become 2- and 3- considered in $\text{PSO}(2, \mathbb{R})$), because the angle between u and v is $\frac{\pi}{2}$ and $\frac{\pi}{3}$, respectively, and $|u| = |v|$. In fact, for a lattice L to have such a rotational symmetry, then we would need $|u| = |v| = r$ and also that the angle between u and v is a rational multiple of π . Note that when $|v| = r$, v lies on the semicircle boundary above, and so we identify the cases θ and $\pi - \theta$ for the angle between u and v . Thus we only need to consider $0 \leq \theta \leq \frac{\pi}{2}$. Also any angle $\theta < \frac{\pi}{3}$ does not actually intersect our region at all, and so doesn't correspond to a lattice (in this case, we would end up with $|v| < |u|$).

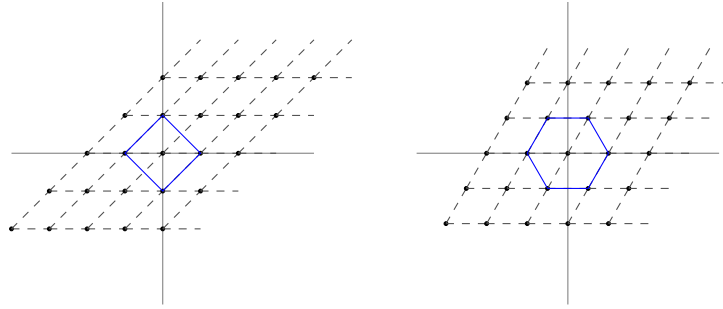


Figure 2: The square and hexagonal lattices

We've already shown the square and hexagonal lattices give us two exceptional fibers, of degrees 2 and 3. Are there any more? It might seem that any rational multiple of π lying between $\frac{\pi}{2}$ and $\frac{\pi}{3}$ would work (we can use $|u| = |v|$ and unimodularity to determine u and v from θ), but this isn't so. For example, look at the lattice when $\theta = \frac{2\pi}{5}$, which does not have rotational symmetry beyond a rotation by π :

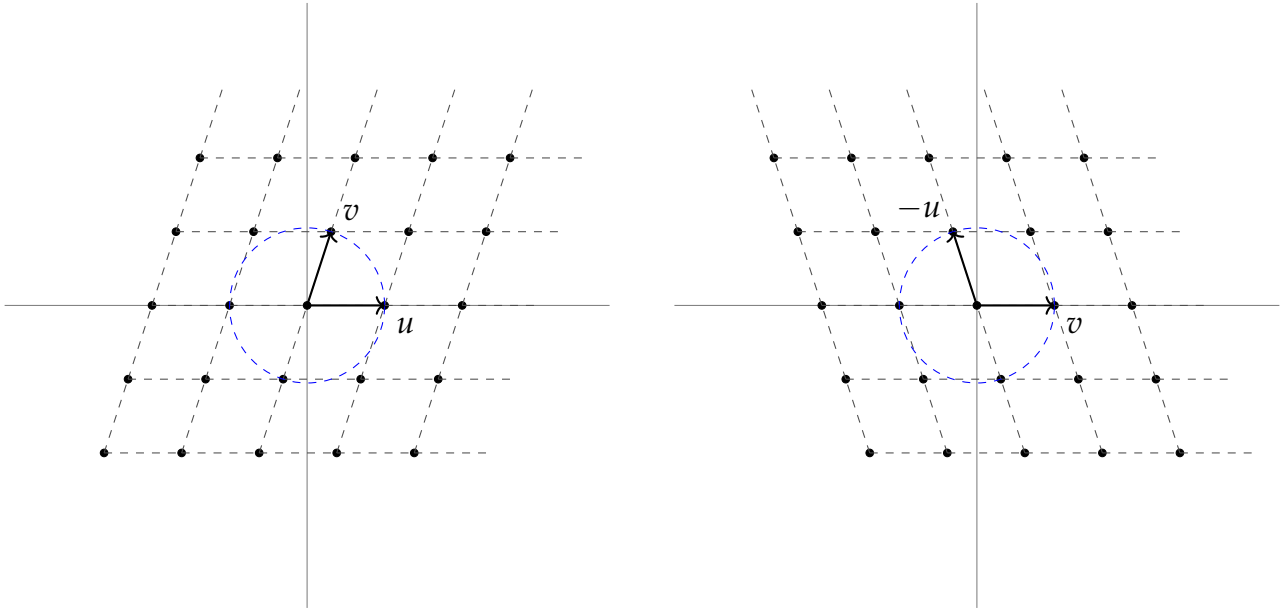


Figure 3: The $\frac{2\pi}{5}$ lattice, rotated

The problem is that we need $\pi - \theta = k\theta$ for $k \in \mathbb{Z}^+$ so that rotation by θ actually fixes the lattice, and gives an exceptional fiber of degree $k + 1$. But this means $\theta = \frac{\pi}{k+1}$, and since $\frac{\pi}{3} \leq \theta \leq \frac{\pi}{2}$, we can only have $k \in \{2, 3\}$. So we've found all exceptional fibers, and have (once more) described the Seifert fibered structure of M .

Bibliography

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