

This is an *old* post of mine, from a now-defunct math blog some of us graduate students kept when I was at Stony Brook. I've cleaned it up a bit and expanded it, in the hopes that it might still be interesting. It is about the **transfer map** of a group; it gives the definitions, and then some applications, including one to manifold theory.

0.1 Definitions

Given an arbitrary group G , and a finite index subgroup H , it is natural to try and study some of the structure of G through asking questions about the structure of H . There are many techniques available for this, but this post will concentrate on one which doesn't seem to be taught in many standard graduate algebra classes, namely *transfer*. Transfer is the study of abelian images of G via abelian images of H . More precisely, if $\phi : H \rightarrow A$ is a group homomorphism to an abelian group A , then we construct a map $\tilde{\phi} : G \rightarrow A$ (I won't call $\tilde{\phi}$ a "lift", since in general we don't have $\tilde{\phi}|_H \equiv \phi$.)

To actually build $\tilde{\phi}$, we first pick a transversal for the cosets of H in G ; that is, a set $T = \{t_1, \dots, t_n\}$ of elements of G ($n = [G : H]$), such that $G = \coprod Ht_i$. Now given any $g \in G$, g acts on the cosets of H , and hence on T ; write this action as $t_i \cdot g$. Then a simple verification shows, for all i , $t_i g (t_i \cdot g)^{-1} \in H$, and so we can (finally) define $\tilde{\phi}$ as

$$\tilde{\phi}(g) = \prod_{i=1}^n \phi(t_i g (t_i \cdot g)^{-1}).$$

Since A is abelian, the order we do this product in doesn't matter. Slightly more work shows that which transversal we choose doesn't matter, and that $\tilde{\phi}$ is a homomorphism.

LEMMA: *If S and T are two transversals for H in G , then both give the same definition of $\tilde{\phi}$.*

Proof. Define a bijection $f : S \rightarrow T$, so that $Hs = Hf(s)$ as cosets. Then because $Hsg = Hf(s)g$, we see that $f(s \cdot g) = f(s) \cdot g$. For convenience, define $h_s = sf(s)^{-1} \in H$. Thus we have

$$\begin{aligned} \tilde{\phi}(g) &= \prod_{t \in T} \phi(tg(t \cdot g)^{-1}) \\ &= \prod_{s \in S} \phi(f(s)g(f(s) \cdot g)^{-1}) \\ &= \prod_{s \in S} \phi(f(s)gf(s \cdot g)^{-1}) \\ &= \prod_{s \in S} \phi(h_s^{-1}sg(s \cdot g)^{-1}h_{s \cdot g}) \\ &= \prod_{s \in S} \phi(h_s^{-1})\phi(sg(s \cdot g)^{-1})\phi(h_{s \cdot g}) \\ &= \prod_{s \in S} \phi(sg(s \cdot g)^{-1}) \end{aligned}$$

where we've used that the target group is abelian, and that g is a permutation on the set S . $\square$

COROLLARY: $\tilde{\phi}$ is a homomorphism.

Proof. If $x \in G$, then both T and $T \cdot x$ are transversals. By the above lemma then, we can write

$$\tilde{\phi}(g) = \prod_{t \in T} \phi((t \cdot x)g(t \cdot xg)^{-1})$$

so that we then have

$$\begin{aligned} \tilde{\phi}(x)\tilde{\phi}(g) &= \prod_{t \in T} \phi(tx(t \cdot x)^{-1}) \prod_{t \in T} \phi((t \cdot x)g(t \cdot xg)^{-1}) \\ &= \prod_{t \in T} \phi(txg(t \cdot xg)^{-1}) \\ &= \tilde{\phi}(xg) \end{aligned}$$

$\square$

0.2 Groups with large centers

Obviously, having $\tilde{\phi}$ be a homomorphism is great, but also being able to choose the transversal we use (say, dependent on the element g) gives a lot of flexibility in evaluating $\tilde{\phi}$. For example, when H is normal in G , we know G/H is a group, and we can pick a transversal to reflect that (with powers of elements, inverses, etc.). More generally, given $g \in G$, we can break up T into its orbit decomposition under g , and pick orbit representatives $s_1, \dots, s_k \in T$ such that $s_i g^{n_i} s_i^{-1} \in H$, where n_i is the size of the orbit of s_i , with $\sum n_i = n$. Then we have

$$\tilde{\phi}(g) = \prod_{i=1}^k \phi(s_i g^{n_i} s_i^{-1}).$$

This observation allows us to prove the following

THEOREM (Schur): *If $Z(G)$ has finite index in G , then G' is finite.*

Proof. Let $\phi : Z(G) \rightarrow Z(G)$ be the identity map, and $T = \{t_1, \dots, t_n\}$ a transversal. As above, for $g \in G$, pick $s_1, \dots, s_k \in T$, along with the corresponding n_i ; then $s_i g^{n_i} s_i^{-1} \in Z(G)$, so $s_i g^{n_i} s_i^{-1} = s_i^{-1} (s_i g^{n_i} s_i^{-1}) s_i = g^{n_i}$, and thus

$$\tilde{\phi}(g) = \prod_{i=1}^k \phi(s_i g^{n_i} s_i^{-1}) = \prod_{i=1}^k \phi(g^{n_i}) = \prod_{i=1}^k g^{n_i} = g^n$$

Since G' is in the kernel of $\tilde{\phi}$, we have the exponent of G' is bounded by n . It isn't hard to see that the commutators $[t_i, t_j]$ generate G' , so it is finitely generated. Then $G' \cap Z(G)$ is a finitely generated abelian group, with bounded exponent, hence finite; since it has finite index in G' , G' is finite. $\square$

I will briefly mention that a similar result holds for higher members of the upper and lower central series of G , but won't say any more about it.

0.3 Groups which are almost free abelian

Schur's theorem allows us to prove a very nice result for torsion-free groups. The non-abelian analog is the very difficult Stallings-Swan theorem; the case below has a very short proof.

THEOREM: *If G is a torsion-free group, with a subgroup $H \leq Z(G)$ such that $[G : H]$ is finite and H is free abelian, then $G \cong H$.*

Proof. Since $[G : Z(G)]$ is finite, Schur's theorem implies G' is finite; since G is torsion-free, $G' = 1$ and G is abelian. Let $n = [G : H]$; then the mapping $f : G \rightarrow H$, sending g to g^n , is a monomorphism, and so $G \cong f(G)$ is free abelian. Rank inequalities imply $G \cong H$. $\square$

Note that if $G = \langle a, b \mid b^{-1}ab = a^{-1} \rangle$ is the fundamental group of the Klein bottle, and $H_1 = \langle a, b^2 \rangle$, $H_2 = \langle b^2 \rangle$, we see that H_1 is free abelian of finite index, and H_2 is free abelian and central; thus neither hypothesis can be dropped from this result.

0.4 Coverings of manifolds

For the final application of transfer, I will prove a result about smooth manifolds. An old comprehensive exam problem asks

If M is a smooth compact manifold, with $\mathbb{Z}/3\mathbb{Z}$ acting freely, and N is the quotient space, show that if $H_{dR}^1(M) = \mathbb{R}$, then $H_{dR}^1(N) \neq 0$.

The result I will prove is more general:

THEOREM: *Let M and N be smooth compact manifolds, and $p : M \rightarrow N$ a regular n -sheeted covering, with n odd. If $H_{dR}^1(M) = \mathbb{R}$, then $H_{dR}^1(N) = \mathbb{R}$.*

Before beginning the proof, I mention that if n is allowed to be even, or $\dim(H_{dR}^1(M)) > 1$, then the result is false.

Proof. We first need to translate this problem into a statement about groups. To do so, we recall two well-known facts from topology:

- if X is a smooth manifold, then $H_{dR}^1(X) = \text{Hom}(\pi_1(X), \mathbb{R})$;
- if Y is a compact manifold, then $\pi_1(Y)$ is finitely generated.

I will also note some easy-to-prove results from group theory:

- if C is a group, then $\text{Hom}(C, \mathbb{R}) = \text{Hom}(C/C', \mathbb{R})$;
- if C is finitely generated, then C/C' is a finitely generated abelian group, and

$$\text{Hom}(C/C', \mathbb{R}) = \mathbb{R}^d$$

where d is the Betti number for C/C' .

So let $G = \pi_1(N)$, $H = \pi_1(M)$; the hypotheses imply that we can treat H as a normal subgroup of G , with $[G : H] = n$. Also from the hypotheses, $H/H' \cong \mathbb{Z} \times B$, where B is some finite abelian group. The subgroup of H corresponding to B (call it K) is such that K is normal in G and $H/K \cong \mathbb{Z}$. We also have $\text{Hom}(G, \mathbb{R}) = \text{Hom}(G/K, \mathbb{R})$ (any homomorphism $\alpha : G \rightarrow \mathbb{R}$, when restricted to H , has K in its kernel). So if we rename some things (to remove clutter), say $L = G/K$ and $J = H/K$, then we have a group L with a normal subgroup J , such that J is infinite cyclic and $[L : J] = n$. We will be done if we can show the Betti number of L/L' is 1.

Now $N_L(J) = L$, and J is abelian, so $J \leq C_L(J)$; since $|\text{Aut}(J)| = 2$ and n is odd, $J \leq Z(L)$ by the normalizer-centralizer theorem. We can thus define the transfer map (as in the proof of Schur's theorem) $\gamma : L \rightarrow J$, with $\gamma(x) = x^n$. Since the kernel of this map is $\text{tor}(L)$ and it has non-trivial image, we have a short exact sequence

$$1 \rightarrow \text{tor}(L) \rightarrow L \rightarrow \mathbb{Z} \rightarrow 1.$$

Since $\mathbb{Z}$ is free, this sequence splits, and we get $L \cong \text{tor}(L) \rtimes \mathbb{Z}$. Now $\text{tor}(L)$ is finite, since the canonical quotient map $L \rightarrow L/J$ is injective on $\text{tor}(L)$, and $|L/J| = n$; we also have $L/\text{tor}(L) \cong \mathbb{Z}$. Thus $L/L' \cong \mathbb{Z} \times D$, with D some finite group, and the proof is complete. $\square$