

Introduction

This note contains a bunch of results about generic topological groups. Recall that a topological group is a topological space G with continuous maps $\mu : G \times G \rightarrow G$ and $\iota : G \rightarrow G$ that, taken as multiplication and inversion, make G a group. We also insist that the identity $\{e\}$ of G is a closed subset in the topology of G .

Preliminaries

To ease notation, we will use the $\cdot$ symbol for the continuous map μ when it doesn't cause confusion. Similarly, we use $^{-1}$ for the continuous inversion map. Inversion, being its own inverse, is of course a homeomorphism.

We note that for any $g \in G$, the map $x \mapsto xg$ is a homeomorphism. This is because it can be written as the composition $x \mapsto \mu(x, g)$ which is the composition of continuous functions, with a continuous inverse $x \mapsto \mu(x, g^{-1})$. Similarly, multiplying on the left by g is a homeomorphism. We will call these homeomorphisms R_g and L_g respectively. Note that R_g and L_g let us move certain "local" properties around e to arbitrary elements of G .

Let U be any open neighborhood around e . Since $e \cdot e = e$, by continuity there are open subsets U_L and U_R around e such that $U_L \cdot U_R \subset U$. If $V = U_L \cap U_R$, then we have $V \cdot V \subset U$. If we write $W = V \cap V^{-1}$, then since $e^{-1} = e$, W is an open neighborhood around e , with $W = W^{-1}$, and $W \cdot W \subset U$. We call W a **symmetric neighborhood** around e . Via R_g (or L_g), every element of G has a basis of symmetric neighborhoods.

Let $H \leq G$ be a subgroup of G . Then the closure $\overline{H}$ is also a subgroup of G . To see this, let $x, y \in \overline{H}$. Then for any open neighborhood U of xy^{-1} , by continuity there are open neighborhoods $x \in V$ and $y \in W$ with $V \cdot W^{-1} \subset U$. Since $x, y \in \overline{H}$, there are elements of H with $h_1 \in V$ and $h_2 \in W$. Then $h_1 h_2^{-1} \in U \cap H$. Since U was arbitrary, $xy^{-1} \in \overline{H}$ and so $\overline{H}$ is a subgroup. Of course, H and $\overline{H}$ are both topological groups with the restrictions of the μ and ι maps.

Now let $N \triangleleft G$ be a normal subgroup. Then $\overline{N}$ is also a normal subgroup: we just saw that it's a subgroup, and for any $g \in G$, $x \mapsto g^{-1}xg$ is a homeomorphism, so commutes with taking the closure. Thus since $g^{-1}Ng = N$, we also have $g^{-1}\overline{N}g = \overline{N}$. We can give G/N the quotient topology from the canonical algebraic map $q : G \rightarrow G/N$. Note this is an open map: the saturation of any open subset $U \subset G$ is $U \cdot N = \bigcup_{n \in N} Un$, a union of open subsets. This implies the product map $q \times q : G \times G \rightarrow G/N \times G/N$ is an open (hence quotient) map, and that allows us to use the defining property of quotient maps to see multiplication

and inversion on G/N are continuous:

$$\begin{array}{ccc} G \times G & \xrightarrow{\mu} & G \\ q \times q \downarrow & & \downarrow q \\ G/N \times G/N & \xrightarrow{\bar{\mu}} & G/N \end{array} \qquad \begin{array}{ccc} G & \xrightarrow{\iota} & G \\ q \downarrow & & \downarrow q \\ G/N & \xrightarrow{\bar{\iota}} & G/N \end{array}$$

When N is a closed normal subgroup, then the identity $\bar{e} \in G/N$ is closed, since $q^{-1}(\bar{e}) = N$. Thus G/N is also a topological group.

Separation axioms

First, note that G is clearly T_1 since for any $g \in G$, $g = L_g(e)$ is the homeomorphic image of a closed subset. It's also not too hard to see G is Hausdorff: if $x \neq y$ are two distinct elements of G , then let U be a symmetric neighborhood of e that does not contain $x^{-1}y$ (this neighborhood exists because $x^{-1}y$ is closed). Then $xU \cap yU = \emptyset$; indeed, if $xu = yv$ for $u, v \in U$, then $x^{-1}y = uv^{-1} \in U \cdot U^{-1} \subset G \setminus \{x^{-1}y\}$ is a contradiction.

A similar argument shows G is regular. If $x \notin C$ for some closed subset $C \subset G$, then there's an open set $x \in U$ disjoint from C . Let V be a symmetric neighborhood around e contained in $x^{-1}U$, so that $xV \subset U$. Then $xV \cap CV = \emptyset$; otherwise, there would be $c \in C$ with $xu = cv$, and thus $c = xuv^{-1} \in xV \cdot V^{-1} \subset U$ is a contradiction. In fact, G is **completely regular**, which means that there exists a continuous function $f : G \rightarrow [0, 1]$ such that $f(x) = 0$ and $f(C) = \{1\}$. Note that complete regularity passes to subspaces and products.

To prove G is completely regular, we follow the key steps of the typical proof of the Urysohn Lemma. Note that if we construct a function f such that $f(e) = 0$ and $f(x^{-1}C) = \{1\}$, then we can use $g \mapsto f(x^{-1}g)$ for our separating function. Thus it suffices to separate e and a closed set C not containing e . Let U_0 be a symmetric open neighborhood separating e from C . Having defined U_n , let U_{n+1} be a symmetric neighborhood of e with respect to U_n , and in particular so that $U_{n+1} \cdot U_{n+1} \subset U_n$. Now let $0 < r \leq 1$ be a dyadic rational, where

$$r = \frac{1}{2^{n_1}} + \frac{1}{2^{n_2}} + \cdots + \frac{1}{2^{n_k}}$$

Here $n_1 > n_2 > \cdots > n_k$. We then define V_r as

$$V_r = U_{n_1} \cdot U_{n_2} \cdots U_{n_k}$$

To extend this to all dyadic rationals, we define $V_r = \emptyset$ when $r \leq 0$ and $V_r = G$ when $r > 1$. The main fact about these V_r that we need is that if we can write $r = \frac{k}{2^n}$ for some integers k and n , then with $s = \frac{k+1}{2^n}$ we have

$$U_n \cdot V_r \subset V_s \tag{1}$$

To see this is true, note it's trivial when $r \leq 0$ or $r \geq 1$. Otherwise consider separately the cases k is even or odd: when $k = 2m$ is even, we have $n_1 < n$ and thus

$$\begin{aligned} U_n \cdot V_r &= U_n \cdot U_{n_1} \cdots U_{n_k} \\ &= V_s \end{aligned}$$

Otherwise, if $k = 2m + 1$ is odd we have $n_1 = n$ and

$$\begin{aligned} U_n \cdot V_r &= U_n \cdot U_n \cdot U_{n_2} \cdots U_{n_k} \\ &\subset U_{n-1} \cdot U_{n_2} \cdots U_{n_k} \end{aligned}$$

But this last set is $U_{n-1} \cdot V_t$ with $t = \frac{m}{2^{n-1}}$, so induction on n shows this is contained in V_s [note $t = s + \frac{1}{2^{n-1}}$].

A consequence of eq. (1) is that if $r < s$ are dyadic rationals, then $V_r \subset V_s$. That's because we can write $r = \frac{k}{2^n}$ and $s = \frac{m}{2^n}$ for some integers k, m, n with $m > k$. Then repeated application of eq. (1) gives

$$U_n^{m-k} \cdot V_r \subset V_s$$

Since $e \in U_n$, the left-hand side contains V_r . We are now ready to define f :

$$f(g) = \inf\{r \mid g \in V_r\} \quad (2)$$

Since $V_0 = \emptyset$, $f(g) \geq 0$; since $V_r = G$ for all $r > 1$, $f(g) \leq 1$. Now $e \in U_n$ for all n , and thus $e \in V_r$ for all $r \geq 0$, so that $f(e) = 0$. If $g \in C$, then $g \notin U_0 = V_1$, and since $V_r \subset V_1$ for all $r < 1$, we see that $f(g) = 1$. All that's left to show then is that f is continuous. In fact, f is uniformly continuous.

Let $\epsilon > 0$ and choose an n such that $\frac{1}{2^n} < \frac{\epsilon}{2}$. We show that for $h \in U_n g$, we have $|f(g) - f(h)| < \epsilon$. Observe that if $h = ug$, then $g = u^{-1}h \in U_n h$ by the symmetry of U_n . There exists $r = \frac{k}{2^n}$ and $s = \frac{k+1}{2^n}$ such that $g \in V_s \setminus V_r$, and we have $0 \leq k \leq 2^n$. Define $q = r - \frac{1}{2^n}$ and $t = s + \frac{1}{2^n}$. Then by eq. (1) we have

$$\begin{aligned} h &\in U_n \cdot V_s \subset V_t \\ g &\notin U_n \cdot V_q \subset V_r \end{aligned}$$

The second line implies that $h \notin V_q$. This means that $q \leq f(h) \leq t$, and together with $r \leq f(g) \leq s$ we finally have

$$|f(g) - f(h)| \leq \frac{2}{2^n} < \epsilon$$

Not all topological groups are normal: the group $\mathbb{Z}^{\mathbb{R}}$ is not normal, but is a topological group with componentwise addition. Below we will see that a locally compact group is paracompact, and paracompact regular spaces are normal. We'll also see a first-countable group is metrizable, and hence also normal.

Compactness and countability

Let G be a topological group that is first countable, meaning there's a countable neighborhood basis around the identity. We then show that being Lindelöf, separable, or second countable are all equivalent for G . Note that if $\{U_n\}$ is a countable basis around e , then we can define a slightly nicer one inductively: let V_1 be a symmetric neighborhood for e in U_1 . Recall this means $V_1^{-1} = V_1$ and $V_1 \cdot V_1 \subset U_1$. Having defined V_n , define V_{n+1} as a symmetric neighborhood of e inside $V_n \cap U_{n+1}$. Then the $\{V_n\}$ are still a countable basis of neighborhoods around e , since each is open and $V_n \subset U_m$ for all $m \leq n$.

Suppose G is Lindelöf, meaning every open cover has a countable subcover. Let $\{V_n\}$ be a countable symmetric basis around e as constructed above. For a fixed n , $\{gV_n\}$ is an open cover of G , hence has a countable subcover $\{g_{in}V_n\}$. The countable set $\{g_{in}\}$ as i and n range over the positive integers is then a dense subset of G . To see this, let U be any open subset of G , containing an element h . There is then an n such that $hV_n \subset U$. There is a $g \in \{g_{in}\}$ with $h \in gV_n$, and by symmetry this means $g \in hV_n \subset U$. Similarly, the countable set $\{g_{in}V_n\}$ is a basis for G . For if we have $hV_n \subset U$ as before, then surely $hV_{n+1} \subset U$. Just as before then, there's a $g \in \{g_{in}\}$ with $g \in hV_{n+1}$. But then

$$h \in gV_{n+1} \subset hV_{n+1} \cdot V_{n+1} \subset hV_n \subset U$$

Since U and h were arbitrary, we see that $\{g_{in}V_n\}$ is a basis and G is second-countable. We have just shown that Lindelöf implies separability, which implies second-countability. Since second-countability always implies Lindelöf for all topological spaces, we have shown the equivalence for first-countable G . We will see below that in fact G is metrizable, and the equivalence of these countability conditions always holds for metric spaces.

Now suppose that G is compact (forget first-countability). A closed normal subgroup N is then also compact, while G/N is compact as a quotient of a compact space. It turns out the converse works as well: if N and G/N are compact, then G is compact too. The key ingredient is that the quotient map $p : G \rightarrow G/N$ is actually a closed map in this case. Assuming that, then p is a **perfect map**, meaning it's a continuous closed surjection where the preimage of any point is compact. For any perfect map $f : X \rightarrow Y$, when Y is compact so is X .

To see that $p : G \rightarrow G/N$ is closed, let $C \subset G$ be a closed subset, and note that since p is a quotient map, we need only to show the saturation CN is closed. Suppose $g \notin CN$. Then $G \setminus C$ is an open subset around gN . Recalling that μ is group multiplication, then $\mu^{-1}(G \setminus C)$ is an open subset containing the product set $\{g\} \times N$. The tube lemma implies the existence of an open subset $U \subset G$ with $U \cdot N \subset G \setminus C$. But then if $u \in U$, we have $uN \cap C = \emptyset$, and thus U is an open set around g that is disjoint from CN , and this shows CN is closed.

Finally for this section, assume G is locally compact. We show below that these groups have nice analytical properties. For now though, we show that G is paracompact. Let V be an

open neighborhood of e with compact closure; we can further refine V to be a symmetric neighborhood with compact closure. The set $H = \bigcup V^n$ is actually a subgroup: it contains the identity since $e \in v$; it contains inverses since V (and hence V^n) is symmetric; and it contains products since $u \in V^m$ and $v \in V^n$ means $uv \in V^{m+n}$. H is clearly open, but then every coset of H is a homeomorphic image of H , and so is also open. Then $G \setminus H$ is a union of open sets, and so H is closed as well. This implies $\overline{V} \subset H$, and since H is closed under products,

$$\bigcup \overline{V}^n \subset H = \bigcup V^n \subset \bigcup \overline{V}^n$$

We thus have equality throughout, and H is the countable union of compact subsets: it is σ -**compact**. The part we care about is that H is Lindelöf, and hence so is every one of its cosets. Let $\{U_i\}$ be an arbitrary open cover of G . We show that there is a countably locally finite refinement of $\{U_i\}$, which for regular spaces like G is equivalent to being paracompact.

Let τ be a set of coset representatives for H in G : $G = \bigsqcup_{\tau} tH$. Write $\{U_n^t\}$ for the countable subcover of $\{U_i\}$ covering the coset tH . If we then define $W_n = \{U_n^t \cap tH \mid t \in \tau\}$, we see that each W_n is an open refinement of our original cover. W_n is also locally finite; even stronger, the elements of W_n are disjoint, since distinct cosets are disjoint. Since $\bigcup W_n = \bigsqcup tH = G$, $\{W_n\}$ is the countably locally finite refinement we were after.

Homotopy

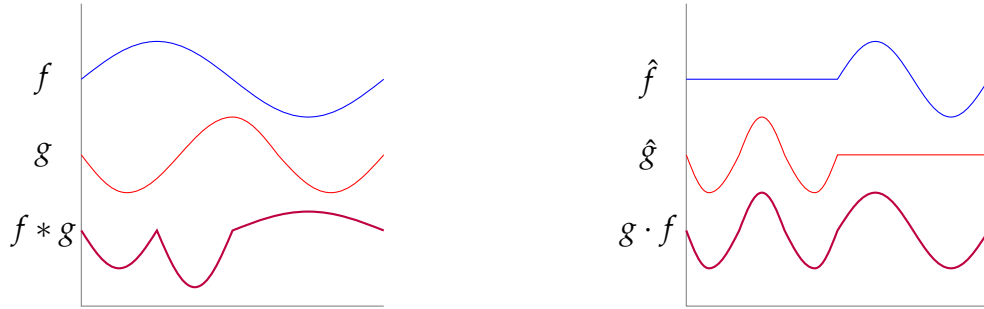
All the homotopy groups we consider in this section will be based at the identity: $\pi_n(G) \equiv \pi_n(G, e)$. We also need to consider the connected component of e in G , which we write as $\mathcal{I}$. This is actually a closed normal subgroup of G : being closed is automatic since $\overline{\mathcal{I}}$ is connected; it is closed under conjugation since $x \mapsto g^{-1}xg$ is a homeomorphism, so $g^{-1}\mathcal{I}g$ is the connected component containing $g^{-1}eg = e$; similarly, $\mathcal{I}^{-1} = \mathcal{I}$ since inversion is a homeomorphism and $e^{-1} = e$; and finally, $\mu(\mathcal{I}, \mathcal{I})$ is the continuous image of the connected set $\mathcal{I} \times \mathcal{I}$, hence is contained in the connected component which contains $e \cdot e = e$, meaning that $\mathcal{I} \cdot \mathcal{I} \subset \mathcal{I}$.

When G is locally path-connected, we see that $\mathcal{I}$ is also the path component of e . In particular, $\pi_0(G)$ can be given a (topological) group structure via the bijection with $G/\mathcal{I}$. This is a bijection since the cosets of $\mathcal{I}$ are each path components of G , they are mutually disjoint, and cover G . Since $\pi_n(G)$ for $n > 0$ is only concerned with maps that end up in $\mathcal{I}$, for the rest of the section we assume G is connected and locally path-connected.

Let's focus on $\pi_1(G)$ first. Suppose $f, g : [0, 1] \rightarrow G$ are loops based at e . Then there are in fact two products that can be placed on these loops:

$$(f \cdot g)(t) = \begin{cases} f(2t) & t \leq \frac{1}{2} \\ g(2t - 1) & t \geq \frac{1}{2} \end{cases} \quad (f * g)(t) = f(t)g(t)$$

The left-hand product is the usual concatenation of paths (with rescaling); the right-hand product is the pointwise product that takes place in G . Doing the multiplication in G is very powerful, because we can "compress" f and/or g so that they start and/or end with some amount of time spent at e , and then do their usual loop at a higher speed:



To be precise, let ℓ be the compression from the left and ρ compression from the right, where:

$$\ell(s, t) = \begin{cases} \frac{t-s}{1-s} & t \geq s \\ 0 & t \leq s \end{cases} \quad \rho(s, t) = \begin{cases} 1 & t \geq s \\ \frac{t}{s} & t \leq s \end{cases}$$

There are some continuity issues along the borders for ℓ and ρ , but we'll ignore those since they don't affect us. Now consider the map

$$F(s, t) = f(\rho(s, t))g(\ell(1 - s, t)) \quad \frac{1}{2} \leq s \leq 1$$

We have that

$$\begin{aligned} F(\tfrac{1}{2}, t) &= (f \cdot g)(t) \\ F(1, t) &= (f * g)(t) \\ F(s, \{0, 1\}) &= \{e\} \end{aligned}$$

And thus F gives a homotopy between $f \cdot g$ and $f * g$. But what happens if we compress the other way around? Specifically, if

$$G(s, t) = f(\ell(s, t))g(\rho(1 - s, t)) \quad 0 \leq s \leq \tfrac{1}{2}$$

We have that

$$\begin{aligned} G(0, t) &= (f * g)(t) \\ G(\tfrac{1}{2}, t) &= (g \cdot f)(t) \\ G(s, \{0, 1\}) &= \{e\} \end{aligned}$$

And thus G gives a homotopy between $f * g$ and $g \cdot f$. Combining these two homotopies we get that $f \cdot g \sim g \cdot f$, and so we have $[f][g] = [g][f] \in \pi_1(G)$; in other words, $\pi_1(G)$ is Abelian.

Let's switch gears and look at covering spaces for G . We will make repeated use of the fact that if $p : (Y, y) \rightarrow (Z, z)$ is a covering, and $f : (X, x) \rightarrow (Z, z)$ is a continuous map, then f has a lift $F : (X, x) \rightarrow (Y, y)$ precisely when $f_*\pi_1(X, x) \subset p_*\pi_1(Y, y)$, and such a lift is unique. So let $p : (H, \hat{e}) \rightarrow (G, e)$ be a covering. Consider the composite map

$$H \times H \xrightarrow{p \times p} G \times G \xrightarrow{\mu} G$$

As proved above, for any $[f], [g] \in \pi_1(G)$, we have $[f][g] = [f * g]$, which is just $\mu_*([f], [g])$. Using $\hat{e}$ as a basepoint for $\pi_1(H)$, we then see the above composite gives

$$\pi_1(H \times H) = \pi_1(H) \times \pi_1(H) \xrightarrow{p_* \times p_*} \pi_1(G) \times \pi_1(G) \xrightarrow{\mu_*} \pi_1(G)$$

Now for $\alpha, \beta \in \pi_1(H)$, we see that

$$\begin{aligned} \mu_*(p_*\alpha, p_*\beta) &= p_*\alpha \cdot p_*\beta \\ &= p_*(\alpha\beta) \\ &\in p_*\pi_1(H) \end{aligned}$$

So we can then lift our $p \times p$ map to a map $\hat{\mu}$ and get the commutative diagram

$$\begin{array}{ccc} H \times H & \xrightarrow{\hat{\mu}} & H \\ \downarrow p \times p & & \downarrow p \\ G \times G & \xrightarrow{\mu} & G \end{array}$$

Note that $\hat{\mu}(\hat{e}, \hat{e}) = \hat{e}$. The map $\hat{\mu}$ gives a product on H , and the diagram shows p commutes with $\hat{\mu}$: $p\hat{\mu}(h, k) = \mu(p(h), p(k))$. So once we see that H is also a topological group, this shows p is a homomorphism. Before we do that, however, we mention that the uniqueness of the lift $\hat{\mu}$ shows H has a unique product structure making p a homomorphism (and as we'll see, with $\hat{e}$ as identity).

Because $p(\hat{e}) = e$, it follows from the above diagram that the map $h \mapsto p\hat{\mu}(h, \hat{e})$ is just p . It has two lifts though: the identity map on H , and $h \mapsto \hat{\mu}(h, \hat{e})$. Since lifts are unique, it must be that for all $h \in H$, $\hat{\mu}(h, \hat{e}) = h$. Similarly, $\hat{\mu}(\hat{e}, h) = h$. This shows $\hat{e}$ can serve as our identity element. Before showing H has inverses, we quickly see that the inversion map $\iota : G \rightarrow G$ induces inversion in $\pi_1(G)$:

$$\begin{aligned} 1 &= \mu_*([f], \iota_*[f]) \\ &= [f] \cdot \iota_*[f] \end{aligned}$$

Now consider the map $h \mapsto \iota p(h)$. This sends $\alpha \in \pi_1(H)$ to $(p_*\alpha)^{-1} = p_*(\alpha^{-1}) \in p_*\pi_1(H)$. This map then has a lift $\hat{\iota} : H \rightarrow H$, which also commutes with p , in that $p\hat{\iota} \equiv \iota p$. But the map $h \mapsto \mu(p(h), \iota p(h))$ is the constant map to e . It has two lifts now: the constant map from H to $\hat{e}$, and the map $h \mapsto \hat{\mu}(h, \hat{\iota}(h))$. By uniqueness, these are the same maps, and so for all $h \in H$,

$$\hat{\mu}(h, \hat{\iota}(h)) = \hat{e}$$

Similar reasoning shows $\hat{\mu}(\hat{i}(h), h) = \hat{e}$ as well, and thus $\hat{i}(h)$ serves as the inverse to h . To finish the fact that H is a topological group, we need to show the product from $\hat{\mu}$ is associative. But the two maps

$$\begin{aligned}(h, k, r) &\mapsto \hat{\mu}(h, \hat{\mu}(k, r)) \\ (h, k, r) &\mapsto \hat{\mu}(\hat{\mu}(h, k), r)\end{aligned}$$

are both lifts of the map $(h, k, r) \mapsto p(h)p(k)p(r)$, which is well-defined because G is associative, and so they are the same map; thus multiplication in H is associative. Before moving on, we mention one more fact: if G is Abelian, then H is as well. That's because for a fixed $k \in H$, the maps $h \mapsto \hat{\mu}(h, k)$ and $h \mapsto \hat{\mu}(k, h)$ both lift the map $h \mapsto p(h)p(k)$, because G is Abelian. Uniqueness then shows that for all $h \in H$, $\hat{\mu}(h, k) = \hat{\mu}(k, h)$. Since k was arbitrary, H is Abelian.

We now know that $p : H \rightarrow G$ is a surjective homomorphism. What is the kernel K ? Since $K = p^{-1}(e)$ and p is a local homeomorphism, K is discrete. Let $h \in H$ be arbitrary, and let $f : [0, 1] \rightarrow H$ be a path from h to $\hat{e}$. For any $k \in K$, this path then gives a path $t \mapsto f(t)^{-1}kf(t)$ from $h^{-1}kh$ to k , and this path's image is in K since it's a normal subgroup. But K is discrete, so the path is constant: $h^{-1}kh = k$. Because h and k were arbitrary, this shows K is a central subgroup of H .

Conversely, if K is a discrete central subgroup of H , then the quotient map $p : H \rightarrow H/K$ is a covering map. Before proving this, we take a detour to show a discrete subgroup of a topological group is closed, so that H/K is in fact a topological group. So let $D \leq G$ be a discrete subgroup of any topological group G . Let $x \notin D$. Choose a neighborhood U around the identity that does not contain x , and such that $U \cap D = \{e\}$, which is possible by discreteness. Refine this to a symmetric neighborhood V . If there are two elements $g, h \in xV \cap D$, then $g^{-1}h \in V^{-1} \cdot V \subset U$, and also in D since it's a subgroup. Thus $g^{-1}h \in U \cap D = \{e\}$, so we only need to worry about a single element $g \in D$ that's in xV . But then $xV \setminus \{g\}$ is an open neighborhood of x disjoint from D .

To see p is a covering map, let U be a neighborhood of $\hat{e}$ with $U \cap K = \{\hat{e}\}$. Now by continuity and local path-connectedness, choose a path-connected neighborhood V of $\hat{e}$ such that $V \cdot V^{-1} \cdot V \cdot V^{-1} \subset U$. If $W = V \cdot V^{-1}$, then W is a path-connected symmetric neighborhood of $\hat{e}$ with $W \cdot W \subset U$. For any $h \in H$, $p(hW)$ is a path-connected open neighborhood of $hK \in H/K$. Its preimage in H is given by $hWK = \bigcup_K hWk$. To finish the proof, we then want to show for each hWk , p is a homeomorphism onto $p(hW)$, and that the hWk are mutually disjoint.

If $hWk_1 \cap hWk_2 \neq \emptyset$, then there are $u, v \in W$ with $huk_1 = hvk_2$, which then implies $k_1k_2^{-1} = u^{-1}v \in W^{-1} \cdot W \subset U$. Since $U \cap K$ only contains the identity, $k_1 = k_2$. Thus the sets $\{hWk \mid k \in K\}$ are mutually disjoint. For any $k \in K$, hWk is an open subset of H , and thus $p : hWk \rightarrow p(hW)$ is open and continuous. It is clearly surjective since $p(hwk) = p(hw)$. To see it's injective, recall that p is a homomorphism, so that if $p(huk) = p(hvk)$ for $u, v \in W$, then $u^{-1}v \in W^{-1} \cdot W \subset U$ is also in K , and hence is the identity, so that $u = v$ and p is indeed injective.

Finally, what about the covering transformations of the quotient $p : H \rightarrow H/K$? These are the homeomorphisms $\phi : H \rightarrow H$ such that $p\phi \equiv p$. They also form a group, sometimes called the **deck transformations**, which we'll denote by $A(H)$. Now for any $k \in K$, the homeomorphism L_k is a deck transformation, since p is a homomorphism and K is the kernel. Because $L_{kl} \equiv L_k L_l$, we have a homomorphism $K \rightarrow A(H)$. For any nontrivial $k \in K$, L_k is fixed-point free, so not the identity map on H ; thus this homomorphism is injective. If $\phi \in A(H)$, then because $p\phi(\hat{e}) = p(\hat{e})$ is the identity, we must have $\phi(\hat{e}) = k \in K$. Then ϕ and L_k agree on the point $\hat{e}$, and are thus the same map; this shows our homomorphism is surjective and thus $K \cong A(H)$.

Metrics

In this section, G will be a first-countable group. We show that G admits a left-invariant metric d , meaning that $d(gx, gy) = d(x, y)$ for all $g \in G$. The idea is similar to one we've already used twice before, when proving G completely regular, and when talking about countability properties for topological groups. As a reminder, let $\{W_n\}$ be a countable basis around e , then inductively define U_1 as a symmetric neighborhood for W_1 and U_{n+1} as a symmetric neighborhood for $U_n \cap W_{n+1}$. We have

$$\cap U_n \subset \cap W_n = \{e\} \subset \cap U_n \quad (3)$$

where the equality follows because $\{W_n\}$ is a basis around e and G is Hausdorff. Then just as before, we define a "Urysohn function" based on the U_n , $f : G \rightarrow [0, 1]$. The details of this function are in the complete regularity proof above: the definition of f is given in [eq. \(2\)](#). We showed there that f is uniformly continuous, in that for any $\epsilon > 0$, there is an N such that for any $x \in G$ and $n \geq N$ we have

$$|f(x) - f(y)| < \epsilon \quad y \in U_n x \quad (4)$$

As shown there, $f(e) = 0$. Conversely, if $f(x) = 0$, then $x \in \cap U_n$ so that $x = e$ [this does not apply to the complete regularity proof, and depends on [eq. \(3\)](#)]. Given f we define the metric as

$$d(x, y) = \sup_{g \in G} |f(x^{-1}g) - f(y^{-1}g)| \quad (5)$$

This definition may seem odd, but we want the g acting on the right so that below we can leverage uniform continuity easily. We use inverses so that left translation ($x \mapsto hx$) becomes a multiplication on the right ($x^{-1}g \mapsto x^{-1}h^{-1}g$). That's why this is left-invariant, since for any $h \in G$, $h^{-1}g$ goes through G just as g does. Symmetry and non-negativity of d are immediate, as well as the fact that $d(x, x) = 0$. Suppose that $d(x, y) = 0$. Then

in particular, $0 = f(e) = f(y^{-1}x)$, and hence $x = y$. The triangle inequality is also trivial, because

$$\begin{aligned} |f(x^{-1}g) - f(z^{-1}g)| &= |f(x^{-1}g) - f(y^{-1}g) + f(y^{-1}g) - f(z^{-1}g)| \\ &\leq |f(x^{-1}g) - f(y^{-1}g)| + |f(y^{-1}g) - f(z^{-1}g)| \\ &\leq \sup_{g \in G} |f(x^{-1}g) - f(y^{-1}g)| + \sup_{g \in G} |f(y^{-1}g) - f(z^{-1}g)| \end{aligned}$$

What remains is to show that d actually generates the topology of G . We need to show each metric ball is open in G , and that these balls form a basis for G . Suppose then that y is in the ϵ ball around x , $B_\epsilon(x)$. Then $d(x, y) = \delta < \epsilon$, and we can choose a $\eta > 0$ with $\delta + \eta < \epsilon$. By the uniform continuity of [eq. \(4\)](#), we can pick an n such that when $z^{-1} \in U_n y^{-1}$, we have $|f(y^{-1}) - f(z^{-1})| < \eta$. Since for all $g \in G$, we have $z^{-1}g \in U_n y^{-1}g$, this inequality holds for all translates and thus $d(y, z) < \eta$. Thus by the triangle inequality $d(x, z) < \epsilon$, and so using the symmetry of U_n , we see that the open neighborhood $yU_n \subset B_\epsilon(x)$, so this ball is a union of open sets and hence open itself.

To see these metric balls form a basis, let x be an element of an open set U . Then the U_n form a basis around e , and hence there's a U_n with $xU_n \subset U$. But if $d(x, y) < \frac{1}{2^n}$, then in particular (with $g = x$ in [eq. \(5\)](#)) $f(y^{-1}x) < \frac{1}{2^n}$. This means that $y^{-1}x \in U_n$, and so by symmetry $y \in xU_n \subset U$. This implies, with $\epsilon = \frac{1}{2^n}$, that $B_\epsilon(x) \subset U$.

Of course, by moving g from the right to the left in [eq. \(5\)](#), we could produce a right-invariant metric. There is then the question of producing a metric that is both left- and right-invariant. Not all topological groups have such a metric, but it turns out to be equivalent to asking that the countable symmetric basis $\{U_n\}$ at e is actually invariant under conjugation: $g^{-1}U_n g = U_n$ for all n and g . To see this is sufficient, note that it implies every V_r is invariant under conjugation. Thus, $x \in V_r$ is equivalent to $g^{-1}xg \in V_r$ for any $g \in G$. This means that the Urysohn function f is also conjugation invariant, and so our left-invariant metric is also right-invariant:

$$\begin{aligned} d(x, y) &= \sup_{g \in G} |f(x^{-1}g) - f(y^{-1}g)| \\ &= \sup_{g \in G} |f(gx^{-1}gg^{-1}) - f(gy^{-1}gg^{-1})| \\ &= \sup_{g \in G} |f(gx^{-1}) - f(gy^{-1})| \end{aligned}$$

Now suppose we have metric d that is both types of invariant. Then it is in fact conjugation invariant:

$$d(g^{-1}xg, g^{-1}yg) = d(xg, yg) = d(x, y)$$

It follows that $g^{-1}B_\epsilon(x)g = B_\epsilon(g^{-1}xg)$, and thus the $\{B_{1/n}(e)\}$ form a conjugation invariant countable basis at the identity.

While not all groups have a biinvariant metric, some do: for example, it's trivial that Abelian and discrete groups do. Also, compact groups do. For let d be the left-invariant metric constructed above, and then define

$$\hat{d}(x, y) = \max_{g \in G} d(xg, yg)$$

$\hat{d}$ is a metric because d is; it is left-invariant because d is; and it is right-invariant because hg runs through G just as g does. Let $\hat{G}$ be G with the topology induced by $\hat{d}$. Then because $d(x, y) \leq \hat{d}(x, y)$, we see that $\hat{B}_\epsilon(x)$ is open in G . Thus the identity map $G \rightarrow \hat{G}$ is a continuous bijection; but G is compact Hausdorff, and so it's a homeomorphism. This means $\hat{d}$ induced the original topology on G .

Measures

In this section, we consider only locally compact topological groups. We show that there is a left-invariant measure defined on the Borel sets of G . Recall that a **measure** is given as a function λ on the Borel subsets of G , with

- $\lambda(S) \in [0, \infty]$ (*non-negative*);
- $\lambda(\emptyset) = 0$ (*ordinary*);
- $\lambda(\cup A_n) = \sum \lambda(A_n)$ for a countable collection $\{A_n\}$ of mutually disjoint measurable subsets (*countably additive*).

In fact, the measure we construct will also be regular, which means:

- $\lambda(K) < \infty$ for $K \subset G$ compact (*compact-finite*);
- $\lambda(U) = \sup\{\lambda(K) \mid K \subset U\}$ for K compact and U open (*inner regular*);
- $\lambda(A) = \inf\{\lambda(U) \mid A \subset U\}$ for a Borel subset A and U open (*outer regular*).

Of course, the measure that assigns 0 to each subset of G is a measure, but we're more interested in a non-zero version, which when left-invariant is called a **Haar measure**. The main idea is fairly straightforward, and starts with the index of a subgroup $H \leq G$, denoted $[G : H]$, which represents the number of left translates gH of H needed to cover G . We extend this to all subsets in G , by saying that $[A : B]$ is the minimum number of left translates of B required to cover A . This is not really well-defined, but when A is compact and U is open, this is a well-defined finite number, since all translates of U cover A , so a minimal finite cover can be chosen.

Sending $A \mapsto [A : B]$ for some fixed set $B \subset G$ doesn't actually provide a measure, since it doesn't satisfy the countable additivity above. We instead will shrink the B to smaller and smaller open sets of G ; the resulting map needs to be normalized so that the shrinking doesn't lead to a measure that is just ∞ on every subset. Specifically, let W be an open

neighborhood of e whose closure is compact (this is where local compactness is used); we will fix W for the rest of this construction. Then for an open subset $U \subset G$ and a compact set $K \subset G$ we define

$$\lambda_U(K) = \frac{[K : U]}{[\overline{W} : U]}$$

This is a quotient of two finite numbers, and the denominator is non-zero since W is non-empty. Still, however, this doesn't satisfy the additivity requirement we have for measures. This is where we need to shrink the U to produce a limiting measure. The key fact is that for all compact K ,

$$\lambda_U(K) \leq [K : W]$$

This follows easily: if m translates of U cover $\overline{W}$ and n translates of W cover K , then mn translates of U cover K . In symbols, $[K : U] \leq [K : W][\overline{W} : U]$. For a fixed compact set K , let $M_K = [K : W]$. Then for every open subset U , we can think of λ_U as a point in a product space of closed intervals:

$$\lambda_U \in \prod_K [0, M_K]$$

It's in this product space we'll find our measure (at least for the compact subsets). For any open subset $V \subset G$, define N_V to be the closure of the subset $\{\lambda_U \mid U \subset V\}$. Because evaluation on a given compact K is just projection from the product space, it is continuous. That tells us a few things about an arbitrary element $\chi \in N_V$, because the λ_U are dense there:

1. $\chi(\emptyset) = 0$ and $\chi(\overline{W}) = 1$ because that's true for all λ_U .
2. $\chi(K) \leq \chi(L)$ for compact sets $K \subset L$. This is because $\lambda_U(K) \leq \lambda_U(L)$ for any open U , since an open cover of L is also one for K .
3. $\chi(K \cup L) \leq \chi(K) + \chi(L)$ for compact sets K and L . Again this follows because the inequality holds for λ_U , because the union of translates of U covering K and covering L then also covers $K \cup L$.
4. χ is left-invariant, because λ_U is: if the translates $\{g_i U\}$ cover K , then $\{hg_i U\}$ cover hK .

Let $U_1, \dots, U_k$ be a finite collection of open sets. Then for $V = \cap U_i$, we have that $\lambda_V \in \cap N_{U_i}$. This means the $\{N_U\}$ satisfy the finite intersection property, and since the product space is compact, there is an element $\lambda \in \cap_U N_U$. This will be the start of our measure. By (1) and (4), λ is left-invariant, non-zero, and ordinary. Since $\lambda \in \prod_K [0, M_K]$, we also see that (on compact sets) λ is non-negative and compact-finite. What's missing then is inner/outer regularity, and countably additivity [and being defined on all Borel subsets].

While λ is not necessarily countably additive, it is finitely additive; this is what we get from it being contained in all N_U . Half of this is (3) and induction, so we prove the other half (modulo induction). Suppose then that K and L are disjoint compact subsets of G . Since

G is Hausdorff, there are open subsets U_K and U_L separating K and L . Since $Ke \subset U_K$, the tube lemma implies there's an open neighborhood V_K of e such that $KV_K \subset U_K$, and similarly for L . Let V be a symmetric neighborhood of e contained in $V_K \cap V_L$. For any $g \in G$, suppose $K \cap L \cap gV \neq \emptyset$. Then from $k = gu$ and $l = gv$ we see that $k^{-1}l = u^{-1}v$, or $ku^{-1} = lv^{-1}$, implying $KV \cap LV \neq \emptyset$, which is false. Thus if the translates $\{g_i V\}$ cover $K \cup L$, then they can be partitioned into those that intersect K and those that intersect L . The first has at least $\lambda_V(K)$ translates, and the second at least $\lambda_V(L)$ translates. This shows $\lambda_V(K \cup L) \geq \lambda_V(K) + \lambda_V(L)$. Since this holds true for all λ_U with $U \subset V$, it also holds true for λ :

$$\lambda(K \cup L) = \lambda(K) + \lambda(L) \quad (K \cap L = \emptyset)$$

We can define an extension of λ to get inner regularity by construction:

$$\lambda_*(U) = \sup_K \{\lambda(K) \mid K \subset U\}$$

Here U is open and we are indexing over all compact subsets of G , where λ is already defined. Note this agrees with λ on compact open subsets (like $\emptyset$), because $\lambda(K) \leq \lambda(L)$ for $K \subset L$ by (2). Similarly, we have $\lambda_*(U) \leq \lambda_*(V)$ for $U \subset V$ since this holds for λ . Because this is an extension of λ , it is still non-negative, ordinary, compact-finite, and non-zero. It is also left-invariant, because $K \subset U$ if and only if $gk \subset gU$. Still though we are missing outer regularity, and countable additivity. If K is contained in the finite union of open sets $\bigcup U_i$, then we have finite subadditivity: let $C_i = K - \bigcup_{j \neq i} U_j$ be compact disjoint subsets; use the Hausdorffness of G to separate them by disjoint open subsets V_i ; then $K_i = K - V_i$ is such that $K_i \subset U_i$ and $K = \bigcup K_i$. In particular, $\lambda_*(U_i) \geq \lambda(K_i)$ by definition. Thus by the finite subadditivity of λ (via (3) and induction) we have

$$\lambda_*(K) = \lambda(K) \leq \sum \lambda(K_i) \leq \sum \lambda_*(U_i) \quad (6)$$

We now can define an outer measure

$$\lambda^*(A) = \inf \{\lambda_*(U) \mid A \subset U\}$$

Here A is an arbitrary subset of G , and U is an open subset. It's trivial to see that, for open subsets U (like $\emptyset$), we have $\lambda^*(U) = \lambda_*(U)$; it all follows from $\lambda_*(U) \leq \lambda_*(V)$ for open subsets $U \subset V$. We also still have $\lambda^*(A) \leq \lambda^*(B)$ for $A \subset B$. λ^* is left-invariant, because λ_* is, and $A \subset U$ if and only if $gA \subset gU$. We know that λ^* is non-negative, ordinary, and inner and outer regular. Because $\lambda^*(K) \geq \lambda(K)$ for compact K , we know λ^* is non-zero; note that it is not necessarily the case that $\lambda^*(K) = \lambda(K)$. λ^* satisfies all our Haar measure properties except perhaps for compact-finiteness and countable additivity.

Below we show λ^* is countably subadditive. This allows us to show λ^* is compact-finite. In particular, $\lambda^*(W) = \lambda_*(W) \leq \lambda_*(\bar{W}) = 1$. But for any compact $K \subset G$, a finite set of left translates of W covers K , so by subadditivity $\lambda^*(K)$ is finite.

We now show λ^* is countably subadditive for a countable collection of arbitrary subsets $\{A_n\}$. The usual arguments show that Borel subsets are Carathéodory-measurable [this leverages the finite additivity of λ], and thus λ^* restricted to them is countably additive; in other words, λ^* suitably restricted is our Haar measure. First assume we have a countable collection of open subsets $\{U_n\}$. Then $U = \bigcup U_n$ is also open, so its outer measure is determined by compact subsets contained within. If $K \subset U$, then there is a finite collection $\{U_i\}$ whose union contains K . Then from eq. (6) we have

$$\lambda_*(K) \leq \sum \lambda_*(U_i) = \sum \lambda^*(U_i) \leq \sum \lambda^*(U_n)$$

Since this holds for all $K \subset U$, we see that $\lambda^*(U) \leq \sum \lambda^*(U_n)$, as desired. For the general case, we can restrict to $\sum \lambda^*(A_n) < \infty$, otherwise there's nothing to show. For each A_n , choose an open subset U_n with $\lambda^*(A_n) \geq \lambda^*(U_n) - \frac{\epsilon}{2^n}$, which is possible by outer regularity. With $A = \bigcup A_n$ and $U = \bigcup U_n$, then

$$\lambda^*(A) \leq \lambda^*(U) \leq \sum \lambda^*(U_n) \leq \sum \lambda^*(A_n) + \epsilon$$

This is true for all $\epsilon > 0$, and hence we have countable subadditivity. We will write λ (although slightly confusing) for our Haar measure of λ^* restricted to Borel subsets, with the further change that we rescale so that $\lambda(\overline{W}) = 1$.

It turns out that, up to scaling, Haar measure is unique. To see this, let ν be any other Haar measure. We will show there is a universal constant C with $\int f d\lambda = C \int f d\nu$ for all $f \in C_0^+(G)$. The Riesz representation theorem then implies $\lambda \equiv C\nu$. To fix notation, $C_0(G)$ is the set of all continuous, compactly supported, complex-valued functions on G ; $C_0^+(G)$ is the subset of all non-negative non-zero functions. Because $C_0^+(G)$ linearly spans $C_0(G)$, to apply the Riesz representation theorem we can restrict attention to $f \in C_0^+(G)$.

Before proceeding, we need to establish some facts about the integral $\int f d\nu$ for $f \in C_0^+(G)$ and ν a Haar measure on G . First, for any $g \in G$, both $f \circ L_g$ and $f \circ R_g$ are in $C_0^+(G)$, since they are compositions of continuous functions, and the supports are just translates of the support of f (hence still compact). In fact we always have $\int f \circ L_g d\nu = \int f d\nu$ for any $g \in G$. This is obvious for characteristic functions since ν is left-invariant, and then the usual progression (characteristic, simple, non-negative,...) shows it is true for all f in $C_0^+(G)$. It is also true that $\int f d\nu > 0$; this is because f is non-zero, so there is an open set U on which f is positive. So we just need to show $\nu(U) > 0$. If not, then since every compact subset K is contained in a finite union of left translates of U , we would have $\nu(K) = 0$ for all compact subsets, and inner regularity then implies $\nu(V) = 0$ for all open subsets, which means ν is the zero measure, a contradiction.

In fact, if for $f \in C_0^+(G)$ we define

$$F_\nu(g) = \int f \circ R_g d\nu$$

then F_ν is continuous. To see this, let K be the support of f . Note that f is essentially uniformly continuous: for an arbitrary $\epsilon > 0$, for each $x \in K$ choose a neighborhood U_x

of the identity (with compact closure) such that for $y \in xU_x$, we have $|f(x) - f(y)| < \epsilon$. Cover K with finitely many of the $x_iU_{x_i}$, and let V be a symmetric neighborhood contained in $\bigcap U_{x_i}$. Then for $x^{-1}y \in V$, we have $|f(x) - f(y)| < 2\epsilon$. Thus,

$$\begin{aligned} |F_\nu(x) - F_\nu(y)| &\leq \int |fR_x - fR_y| d\nu \\ &\leq \int_{KV} 2\epsilon d\nu & ((gx)^{-1}(gy) = x^{-1}y) \\ &= 2\epsilon \nu(KV) \end{aligned}$$

The integral above can be restricted to KV since $f(gx) = 0$ unless $g \in Kx^{-1} \subset KV$ (similarly for $f(gy)$). Since $\bar{V}$ is compact, $\nu(KV) \leq \nu(K\bar{V})$ is finite, and so we can make the right-hand side above as small as we like, implying F_ν is continuous as stated.

As was mentioned previously, F_ν is positive and continuous. Fix $j \in C_0^+(G)$; for example, use the complete regularity of G to let j be a separating function between e and $G \setminus W$. For $f \in C_0^+(G)$ we then define

$$\hat{f}_\nu(g) = \frac{f(g^{-1})}{J_\nu(g^{-1})}$$

This is well-defined, continuous, and is in $C_0^+(G)$ because f is. The last ingredient of the uniqueness proof is leveraging Fubini's theorem. Let $H \in C_0(G \times G)$. Then using the left-invariance of the integral, we have

$$\begin{aligned} \int \int H(x, y) d\lambda(x) d\nu(y) &= \int \int H(y^{-1}x, y) d\lambda(x) d\nu(y) & (x \mapsto y^{-1}x) \\ &= \int \int H(y^{-1}x, y) d\nu(y) d\lambda(x) \\ &= \int \int H(y^{-1}, xy) d\nu(y) d\lambda(x) & (y \mapsto xy) \end{aligned}$$

We'll call this the "Fubini trick". We can now show that

$$\begin{aligned} \int f d\lambda &= \int f(x) \frac{J_\nu(x)}{J_\nu(x)} d\lambda(x) \\ &= \int \int \frac{f(x)j(yx)}{J_\nu(x)} d\nu(y) d\lambda(x) \\ &= \int \int \frac{f(y^{-1})j(x)}{J_\nu(y^{-1})} d\lambda(x) d\nu(y) & (\text{Fubini trick}) \\ &= J_\lambda(e) \int \hat{f} d\nu \end{aligned}$$

Succinctly, $F_\lambda(e) = J_\lambda(e) \hat{F}_\nu(e)$. We can exploit this relationship between two Haar measures:

$$\begin{aligned} F_\nu(e) &= J_\nu(e) \hat{F}_\lambda(e) & (\nu \mapsto \lambda) \\ F_\lambda(e) &= J_\lambda(e) \hat{F}_\nu(e) & (\lambda \mapsto \nu) \\ &= \frac{J_\lambda(e)}{J_\nu(e)} F_\nu(e) \end{aligned}$$

So taking $C = J_\lambda(e)/J_\nu(e)$, we see that this works for any $f \in C_0^+(G)$, and we have finished our proof that $\lambda \equiv C\nu$.

The development above can be carried out for right-invariant Haar measures as well, or one can be defined as $\rho(A) = \lambda(A^{-1})$. Unfortunately, as for metrics, not all groups have biinvariant Haar measures, although they do exist trivially for Abelian groups, as well as discrete groups (in this case Haar measure is the counting measure). Unlike for metrics, however, we can quantify how far λ is from being right-invariant. Define the **modular function**

$$\Lambda(g) = \lambda(\overline{W}g)$$

For all $g \in G$, $0 < \Lambda(g) < \infty$, because $\overline{W}g$ is compact and contains the open subset Wg . In fact, because Haar measure is unique up to scaling, and because $\nu(A) = \lambda(Ag)$ is a Haar measure, we see that

$$\Lambda(g) = \frac{\lambda(Ag)}{\lambda(A)}$$

for any Borel subset A with finite nonzero measure. This is the key to showing Λ is a continuous homomorphism into $\mathbb{R}^+$:

$$\begin{aligned} \Lambda(gh) &= \lambda(\overline{W}gh) \\ &= \lambda(\overline{W}g) \frac{\lambda(\overline{W}gh)}{\lambda(\overline{W}g)} \\ &= \Lambda(g)\Lambda(h) \end{aligned}$$

Now let $\epsilon > 0$ and via outer regularity find an open subset U containing $\overline{W}$ with $\lambda(U) < 1 + \epsilon$. The tube lemma gives the existence of a symmetric neighborhood V of e with $\overline{W}V \subset U$. For $g \in V$, we see that $\overline{W}g \subset U$, and hence $\Lambda(g) < 1 + \epsilon$. Because $g^{-1} \in V$ as well, we have $\overline{W} \subset Ug$ and so

$$\Lambda(g) = \frac{\lambda(Ug)}{\lambda(U)} \geq \frac{1}{1 + \epsilon}$$

Thus we have shown Λ is continuous at e , and since it's a homomorphism ($\Lambda(gV) = \Lambda(g)\Lambda(V)$), it is continuous everywhere. The modular function is important because λ is right-invariant precisely when Λ is the trivial homomorphism. For example, suppose G is compact. Then if $\Lambda(g) > 1$, we have $\Lambda(g^n)$ increasing without bound, contradicting the fact G is compact. Similarly, if $\Lambda(g) < 1$, we can look at $\Lambda(g^{-1}) > 1$. Thus Λ is constant for compact G , and so λ is a biinvariant Haar measure.