

Introduction

This note proves some foundational aspects of group homology, and establishes a very important exact sequence related to low-dimensional homology for a group and its quotient. The final section will contain a few applications of this sequence. The sequence comes from [Sta65], but we follow the proof laid out in [ES70], with some improvements along the way.

Tensors and Tors

Throughout this note, A and B will denote arbitrary (noncommutative) rings with identity. We also insist that all ring homomorphisms are unital (take 1_A to 1_B). We will write ${}_A\mathfrak{M}$ for the collection of left A modules, and similarly $\mathfrak{M}_A$ for the collection of right A modules. All modules will have unital action: $1_A m = m$ for all $m \in M$.

It's useful to think of a module M as both a left module and a right module, which is always possible because modules are Abelian groups, and hence always $\mathbb{Z}$ modules on the left and right. For example, we could write ${}_A\mathfrak{M}$ as ${}_A\mathfrak{M}_{\mathbb{Z}}$ instead. This allows us to consider the **tensor product** of two modules. Let $M \in {}_A\mathfrak{M}_C$ and $N \in {}_C\mathfrak{M}_B$. Then $M \otimes_C N$ is a module in ${}_A\mathfrak{M}_B$.

The tensor product is universal, and in particular maps out of the tensor product always come from **balanced** bilinear maps on the product. Namely, a bilinear map $\phi : M \times N \rightarrow P$ is balanced when $\phi(mc, n) = \phi(m, cn)$ for all $c \in C$. This then induces a unique map $\hat{\phi} : M \otimes_C N \rightarrow P$.

Tensor product is a right exact (covariant) functor. The (left) derived functors of the tensor product are the Tor groups. They measure the inexactness of tensoring an exact sequence. That is, if

$$0 \rightarrow L \rightarrow P \rightarrow M \rightarrow 0$$

is an exact sequence over $\mathfrak{M}_A$, and $N \in {}_A\mathfrak{M}$, then we get a long exact sequence

$$\cdots \rightarrow \mathrm{Tor}_n^A(L, N) \rightarrow \mathrm{Tor}_n^A(P, N) \rightarrow \mathrm{Tor}_n^A(M, N) \rightarrow \mathrm{Tor}_{n-1}^A(L, N) \rightarrow \cdots \quad (1)$$

This sequence terminates in the tensor products:

$$\mathrm{Tor}_1^A(L, N) \rightarrow \mathrm{Tor}_1^A(P, N) \rightarrow \mathrm{Tor}_1^A(M, N) \rightarrow L \otimes_A N \rightarrow P \otimes_A N \rightarrow M \otimes_A N \rightarrow 0 \quad (2)$$

Computing the Tor groups can be done via a projective resolution. If

$$\cdots \rightarrow P_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0 \quad (3)$$

is a projective resolution via right A modules, then after tensoring with $N \in {}_A\mathfrak{M}$, we get a sequence

$$\cdots \rightarrow P_1 \otimes_A N \rightarrow P_0 \otimes_A N \rightarrow 0 \quad (4)$$

The homology of this sequence at the $P_n \otimes_A N$ term gives the group $\text{Tor}_n^A(M, N)$. This is also works if we start with a projective resolution of the left module N instead.

Tensoring with a projective module is exact, and thus in the long exact sequence above with N a projective A module, the sequence remains exact, and thus we have $\text{Tor}_n^A(M, N) = 0$ for all $n > 0$ (note that we always have $\text{Tor}_0^A(M, N) = M \otimes_A N$). This fact allows us to prove a useful isomorphism.

PROPOSITION 1: *If M is a left A module, and $I \leq A$ a (two-sided) ideal, then $(A/I) \otimes_A M \cong M/IM$.*

Proof. Define a bilinear balanced map $\phi : (A/I) \times M \rightarrow M/IM$, given by $\phi(a + I, m) = am$. It's trivial this is well-defined, bilinear, balanced, and respects the left A multiplication on A/I . We thus get an induced map on left A modules $\hat{\phi} : (A/I) \otimes_A M \rightarrow M/IM$. Define a potential inverse map as

$$\psi(m + IM) = (1 + I) \otimes_A m$$

This also respects the left A multiplication, since balance implies

$$\psi(am + IM) = (1 + I) \otimes_A am = (a + I) \otimes_A m = a\psi(m + IM)$$

We are then done if we can show these maps are inverses of one another. But

$$\begin{aligned} \hat{\phi}\psi(m + IM) &= \hat{\phi}((1 + I) \otimes_A m) \\ &= \phi(1 + I, m) \\ &= m \\ \psi\hat{\phi}((1 + I) \otimes_A m) &= \psi(m) \\ &= (1 + I) \otimes_A m \end{aligned}$$

It is enough then to conclude that every element of $(A/I) \otimes_A M$ can be written as $(1 + I) \otimes_A m$. But this is true by bilinearity and the middle equality in the balance derivation above. $\square$

Let's briefly specialize to a group ring $R = \mathbb{Z}[G]$, and let M be a left R module. There is a natural **augmentation map** $\epsilon : R \rightarrow \mathbb{Z}$ given by $\sum a_i g_i \mapsto \sum a_i$. The kernel of this homomorphism is known as the **augmentation ideal** I . In particular, in ${}_R\mathfrak{M}_R$, we have $\mathbb{Z} \cong R/I$. Here $\mathbb{Z}$ (sometimes written as $\mathbb{Z}_R$) is a trivial module over G , since every element $g \in G$ acts trivially; this is because $g - 1 \in I$. We define group homology over M via this trivial module, and the Tor functors:

$$H_n(G, M) = \text{Tor}_n^R(\mathbb{Z}, M) \quad (5)$$

Using [Proposition 1](#) above, we can already prove something about the first homology of a group.

THEOREM 2: *For any group G , we have the isomorphism $H_1(G, \mathbb{Z}) \cong \overline{G}$, where $\overline{G}$ is used to denote the Abelianization of G .*

Proof. We start with the short exact sequence coming from the augmentation map:

$$0 \rightarrow I \rightarrow R \rightarrow \mathbb{Z} \rightarrow 0$$

[Recall that $R = \mathbb{Z}[G]$ is the group ring, and I the augmentation ideal.] We now tensor this on the left with the trivial module $\mathbb{Z}$ to get the long exact sequence

$$\cdots \rightarrow \mathrm{Tor}_1^R(\mathbb{Z}, R) \rightarrow \mathrm{Tor}_1^R(\mathbb{Z}, \mathbb{Z}) \rightarrow \mathbb{Z} \otimes_R I \rightarrow \mathbb{Z} \otimes_R R \rightarrow \mathbb{Z} \otimes_R \mathbb{Z} \rightarrow 0$$

Recalling that $\mathbb{Z} \cong R/I$, [Proposition 1](#) gives the following homomorphisms:

$$\begin{aligned}\mathbb{Z} \otimes_R \mathbb{Z} &\cong \mathbb{Z} \\ \mathbb{Z} \otimes_R R &\cong \mathbb{Z} \\ \mathbb{Z} \otimes_R I &\cong I/I^2\end{aligned}$$

Since the rightmost two terms are both $\mathbb{Z}$, and the map between them is surjective, it is injective as well (every nontrivial quotient of $\mathbb{Z}$ is finite). Also $\mathrm{Tor}_1^R(\mathbb{Z}, R) = 0$ since R is projective over itself. Exactness, together with the definition of group homology above, then gives

$$0 \rightarrow H_1(G, \mathbb{Z}) \rightarrow I/I^2 \rightarrow 0$$

Thus the following lemma completes the proof. □

LEMMA 3: *For the augmentation ideal I of the group ring $\mathbb{Z}[G]$, there is an Abelian group homomorphism $I/I^2 \cong \overline{G}$.*

Proof. Define the map $\alpha : G \rightarrow I/I^2$, given by sending g to $g - 1$. To see this is a group homomorphism, we have

$$\begin{aligned}\alpha(gh) &= gh - 1 \\ &= (g - 1)(h - 1) + (g - 1) + (h - 1) \\ &= (g - 1) + (h - 1)\end{aligned}$$

where the last equality is because $(g - 1)(h - 1) \in I^2$. Thus α induces a group homomorphism $\overline{G}$ to I/I^2 .

The reverse map from I/I^2 to $\overline{G}$ can be defined as

$$\beta\left(\sum a_i g_i\right) = \prod g_i^{a_i}$$

Because $\overline{G}$ is Abelian, we have that $\beta(gh - g - h) = 0$. That allows us to show I^2 is in the kernel of β :

$$\begin{aligned}\beta\left(\sum a_i g_i \sum b_j h_j\right) &= \beta\left(\sum_{i,j} a_i b_j g_i h_j\right) \\ &= \beta\left(\sum_{i,j} a_i b_j (g_i + h_j)\right) & (\beta(gh) = \beta(g + h))\end{aligned}$$

The coefficient of g_i in the above sum is $a_i \sum b_j = 0$, because that sum is zero since we're in the augmentation ideal. Similarly, the coefficient of each h_j is 0, and hence the above is $\beta(0) = 0$. Thus β induces a map from I/I^2 to $\overline{G}$. These are inverses of one another, and so we're done:

$$\begin{aligned}\beta\alpha(g) &= \beta(g - 1) \\ &= g \\ \alpha\beta\left(\sum a_i g_i\right) &= \alpha\left(\prod g_i^{a_i}\right) \\ &= \sum a_i g_i - \sum a_i \\ &= \sum a_i g_i\end{aligned}$$

The last equality follows because $\sum a_i = 0$. □

The proof here can be shortened a bit, once we notice that the augmentation ideal is generated over $\mathbb{Z}[G]$ by elements of the form $g - 1$ for $g \in G$. This implies, for example, that I^2 is generated by elements of the form $(g - 1)(h - 1)$. In fact, this observation allows a fairly concise proof of a generalization of [Lemma 3](#):

LEMMA 4: *Let $N \triangleleft G$ be a normal subgroup, I the augmentation ideal of $\mathbb{Z}[G]$, and K the kernel of the ring map $\mathbb{Z}[G] \rightarrow \mathbb{Z}[G/N]$. Then $\overline{N} \cong K/IK$ as right $\mathbb{Z}[G/N]$ modules.*

Proof. As before, we define a map $\alpha : \overline{N} \rightarrow K/IK$ by sending n to $n - 1$. The same proof as in [Lemma 3](#) shows this is a group homomorphism. To show this is a module homomorphism, we need to define a right G action on $\overline{N}$ and K/IK , and show N acts trivially. The action of $g \in G$ on $\overline{N}$ will be given by $n \mapsto n^g$, and clearly this is trivial when $g \in N$, since $\overline{N}$ is Abelian. The action on K/IK will be given by $k \mapsto kg$, the product taking place in $\mathbb{Z}[G]$. Since $K \subset I$, we see that $k(n - 1) \in IK$, and thus n acts trivially on K/IK as desired. Now, we have

$$\begin{aligned}\alpha(n^g) &= g^{-1}ng - 1 \\ &= (n - 1)g + (g^{-1} - 1)(ng - g) \\ &= (n - 1)g & ((g^{-1} - 1)(ng - g) \in IK)\end{aligned}$$

Thus α is a module homomorphism. To define the inverse β , we note that I is generated over $\mathbb{Z}[G]$ by elements of the form $g - 1$, and K is generated as a right $\mathbb{Z}[G/N]$ module by elements of the form $n - 1$ for $n \in N$ (this depends crucially on $(n - 1)g = g(n^g - 1)$). We set $\beta(n - 1) = n$; the calculation just above shows this is also a module homomorphism (just read it in reverse).

To finish the proof, we need to show IK is in the kernel of β . We have

$$\begin{aligned} (h - 1)(n - 1) &= hn - h - n + 1 \\ &= (hnh^{-1} - 1)h - (n - 1) \\ \beta((h - 1)(n - 1)) &= (hnh^{-1})^h n^{-1} \\ &= 0 \end{aligned}$$

□

Note that in the case $N = G$ above, we get [Lemma 3](#). A different proof of this improved lemma will be given in a later section.

Changing Rings

The material in this section is largely based on chapter IV of [\[HS71\]](#). Let $f : A \rightarrow B$ be a ring homomorphism. Then every left B module N can be made a left A module, by defining $a \cdot n = f(a)n$. We'll write ${}_A N$ to denote this A module. Conversely, if M is a left A module, then we can make a left B module via $B \otimes_A M$, where balance implies $b \otimes am = bf(a) \otimes m$. In particular, if f is surjective, then every tensor can be written as $1 \otimes_A m$. Obviously all of this works essentially unchanged for right modules as well.

Suppose we have this map $f : A \rightarrow B$, and a projective resolution as in [Equation \(3\)](#) for the left A module M . We can tensor with a right B module N in roughly two different ways:

$$\begin{aligned} \cdots &\longrightarrow N_A \otimes_A P_1 \longrightarrow N_A \otimes_A P_0 \longrightarrow 0 \\ \cdots &\longrightarrow N \otimes_B B \otimes_A P_1 \longrightarrow N \otimes_B B \otimes_A P_0 \longrightarrow 0 \end{aligned} \tag{6}$$

Note that the terms of the two sequences are the same, since

$$(N \otimes_B B) \otimes_A M = N \otimes_A M = N_A \otimes_A M$$

The homology of the top sequence gives (by definition) $\text{Tor}_n^A(N_A, M)$. What can we say about the homology of the bottom sequence? First, we need to ensure the right halves are projective:

PROPOSITION 5: For a projective module $P \in {}_A\mathfrak{M}$, the left B module $B \otimes_A P$ is also projective.

Proof. Because P is projective, we can write $P \oplus Q \cong \bigoplus_{\Lambda} A$ for some other left A module Q , and some indexing set Λ . But tensor products commute with direct sums, so we get

$$(B \otimes_A P) \oplus (B \otimes_A Q) \cong \bigoplus_{\Lambda} (B \otimes_A A) \cong \bigoplus_{\Lambda} B$$

So that $B \otimes_A P$ is a projective left B module. □

So the sequence

$$\cdots \rightarrow B \otimes_A P_1 \rightarrow B \otimes_A P_0 \rightarrow B \otimes_A M \rightarrow 0 \quad (7)$$

is still a projective sequence; however, it is not necessarily exact, and thus is not necessarily a projective resolution. But if B is a projective A module, then tensoring with B is exact, and so this is a projective resolution. In other words, when B is a projective A module, this sequence can be used to compute the groups $\mathrm{Tor}_n^B(N, B \otimes_A M)$. Because the terms in each sequence of Equation (6) are equal, we thus see that their homology is the same, and we've proved

LEMMA 6: When B is a projective A module, then $\mathrm{Tor}_n^A(N_A, M) \cong \mathrm{Tor}_n^B(N, B \otimes_A M)$.

In the general case (B not projective over A), we can still say something. We can use the fact each $B \otimes_A P_n$ is projective to inductively lift the identity map on $B \otimes_A M$ to get a commutative diagram

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & B \otimes_A P_n & \longrightarrow & B \otimes_A P_{n-1} & \longrightarrow & \cdots & \longrightarrow & B \otimes_A P_0 & \longrightarrow & B \otimes_A M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & L_n & \longrightarrow & L_{n-1} & \longrightarrow & \cdots & \longrightarrow & L_0 & \longrightarrow & B \otimes_A M & \longrightarrow & 0 \end{array} \quad (8)$$

Here the bottom sequence is a projective resolution (over B) of $B \otimes_A M$; since it is exact, the induced map on homology is well-defined. Thus tensoring both sequences with N and taking homology (and remembering the equality of sequences in Equation (6)) shows

LEMMA 7: Given a ring homomorphism $f : A \rightarrow B$, there are well-defined maps $f_n : \mathrm{Tor}_n^A(N_A, M) \rightarrow \mathrm{Tor}_n^B(N, B \otimes_A M)$.

We'll see later that the mere existence of this map can tell us something useful in the right setting.

Let's again specialize to group rings for the moment. Let $R = \mathbb{Z}[G]$ for some group G , and recall that $\mathbb{Z}$ can be treated as the trivial module, isomorphic to R/I , with I the augmentation ideal. Changing the ring to $\mathbb{Z}$ in this case has a well-defined meaning:

PROPOSITION 8: *If M is a left R module for the group ring $R = \mathbb{Z}[G]$, then $\mathbb{Z} \otimes_R M \cong M_G$, the largest quotient of M on which G acts trivially. Similarly for right modules.*

Proof. We know from [Proposition 1](#) that $\mathbb{Z} \otimes_R M \cong M/IM$. Because $g - 1 \in I$ for every $g \in G$, we see that G acts trivially on M/IM . Conversely, if G acts trivially on M/N for some submodule $N \leq M$, then $(g - 1)M \subset N$ for every $g \in G$, and so $IM \leq N$. Thus $M_G = M/IM$. The right module case is essentially identical to prove. $\square$

Exact Sequence

In this section, we focus on the group ring case. We will use $R = \mathbb{Z}[G]$ as the group ring for an arbitrary group, $N \triangleleft G$ a normal subgroup, and $S = \mathbb{Z}[H]$ the group ring over the quotient $H = G/N$. Labeling K as the kernel of the ring homomorphism $R \rightarrow S$, we focus on the short exact sequence

$$0 \rightarrow K/IK \rightarrow I/IK \rightarrow I/K \rightarrow 0 \quad (9)$$

Recall that $I \leq R$ is the augmentation ideal; I/K is the augmentation ideal of S , which we will call J in a few places. Noting that $S = R/K$ and using [Proposition 1](#), we can also write this sequence as

$$0 \rightarrow K/IK \rightarrow I \otimes_R S \rightarrow J \rightarrow 0 \quad (10)$$

For a given left S module M , we can tensor and get the long exact sequence

$$\cdots \rightarrow \text{Tor}_1^S(I \otimes_R S, M) \rightarrow \text{Tor}(J, M) \rightarrow (K/IK) \otimes_S M \rightarrow I \otimes_R S \otimes_S M \rightarrow J \otimes_S M \rightarrow 0 \quad (11)$$

Our goal is to express each term in the above sequence via group homology. We'll do that with a series of short lemmas.

LEMMA 9: *For any group ring $A = \mathbb{Z}[F]$ with augmentation ideal U , there is an isomorphism $\text{Tor}_n^A(U, M) \cong H_{n+1}(F, M)$ for $n \geq 1$. Moreover, we can identify $H_1(F, M)$ with the kernel of the multiplication map $U \otimes_A M \rightarrow M$.*

Proof. We tensor the short exact sequence

$$0 \rightarrow U \rightarrow A \rightarrow \mathbb{Z} \rightarrow 0$$

with the left module M to get the long exact sequence

$$\cdots \rightarrow \text{Tor}_n^A(A, M) \rightarrow \text{Tor}_n^A(\mathbb{Z}, M) \rightarrow \text{Tor}_{n-1}^A(U, M) \rightarrow \text{Tor}_{n-1}^A(A, M) \rightarrow \cdots$$

The terms on the end are 0 because A is free over itself, and thus by the definition of group homology we get our isomorphism. If we look at the end of this long exact sequence, we have (with $\text{Tor}_1^A(A, M) \cong 0$)

$$0 \rightarrow \text{Tor}_1^A(\mathbb{Z}, M) \rightarrow U \otimes_A M \rightarrow A \otimes_A M \rightarrow \mathbb{Z} \otimes_A M \rightarrow 0$$

Exactness implies that $\text{Tor}_1^A(\mathbb{Z}, M) = H_1(F, M)$ is the kernel of the map $U \otimes_A M \rightarrow A \otimes_A M$, and this last term is isomorphic to M via $a \otimes m \mapsto am$. $\square$

LEMMA 10: *There is a surjective map from $H_2(G, M)$ to $\text{Tor}_1^S(I \otimes_R S, M)$.*

Proof. Let $0 \rightarrow L \rightarrow P \rightarrow M \rightarrow 0$ be the start of a projective resolution (over S) for M ; in particular, P is projective. Note this means that $\text{Tor}_1^S(I \otimes_R S, P) = 0$. Just as in Equation (6), we get two sequences:

$$\begin{array}{ccccccccc} \text{Tor}_1^R(I, P) & \longrightarrow & \text{Tor}_1^R(I, M) & \longrightarrow & I \otimes_R L & \longrightarrow & I \otimes_R P & \longrightarrow & I \otimes_R M \\ \downarrow & & \downarrow f_1 & & \downarrow = & & \downarrow = & & \downarrow = \\ 0 & \longrightarrow & \text{Tor}_1^S(I \otimes_R S, M) & \longrightarrow & I \otimes_R S \otimes_S L & \longrightarrow & I \otimes_R S \otimes_S P & \longrightarrow & I \otimes_R S \otimes_S M \end{array}$$

Here the equality over the tensor products is from the discussion following Equation (6), and f_1 is the map of Lemma 7 (everything here is instead using right modules, but the rest remains the same). By the five lemma, f_1 is surjective. By Lemma 9, we have an isomorphism $\text{Tor}_1^R(I, M) \cong H_2(G, M)$. $\square$

We can now also give the other proof of Lemma 4.

LEMMA 11: *K/IK is isomorphic as a right S module to $\overline{N}$.*

Proof. We start with the short exact sequence $0 \rightarrow I \rightarrow R \rightarrow \mathbb{Z} \rightarrow 0$, and tensor with S :

$$0 \rightarrow \text{Tor}_1^R(\mathbb{Z}, S) \rightarrow I \otimes_R S \rightarrow S \rightarrow \mathbb{Z} \otimes_R S \rightarrow 0$$

Thus $\text{Tor}_1^R(\mathbb{Z}, S)$ is isomorphic to the kernel of the map $I \otimes_R S \rightarrow S$. But by Proposition 1

$$I \otimes_R S \cong I \otimes_R R/K \cong I/IK$$

And since $S \cong R/K$, this kernel is by Equation (9) isomorphic to K/IK . So it's enough to show $\text{Tor}_1^R(\mathbb{Z}, S)$ is isomorphic to $\overline{N}$. But R is projective over $T = \mathbb{Z}[N]$; indeed, for any transversal Γ of N in G , we have $R \cong \bigoplus_{\Gamma} T$. Thus we have

$$\begin{aligned} \text{Tor}_1^R(\mathbb{Z}, S) &\cong \text{Tor}_1^R(\mathbb{Z}, R \otimes_T \mathbb{Z}) && (\text{Proposition 8}) \\ &\cong \text{Tor}_1^T(\mathbb{Z}, \mathbb{Z}) && (\text{Lemma 6}) \\ &= H_1(N, \mathbb{Z}) \\ &\cong \overline{N} && (\text{Theorem 2}) \end{aligned}$$

$\square$

The final piece we need is to show that we can "branch" our exact sequence [Equation \(11\)](#) to land in a submodule of $I \otimes_R M$. This is generic, and so we present it here as a lemma:

LEMMA 12: *If $X \xrightarrow{g} Y \xrightarrow{h} Z \rightarrow 0$ is exact, and $V \leq Y$ is a submodule of Y containing $g(X)$, then so is*

$$X \rightarrow V \rightarrow h(V) \rightarrow 0$$

Proof. Clearly the map $V \rightarrow h(V)$ is onto; its kernel is given by $g(X) \cap V = g(X)$. $\square$

LEMMA 13: *The map $K/IK \otimes_S M \rightarrow I \otimes_R S \otimes_S M$ lands inside an S submodule isomorphic to $H_1(G, M)$.*

Proof. Consider the sequence

$$K/IK \otimes_S M \rightarrow I \otimes_R S \otimes_S M \cong I \otimes_R M \rightarrow M$$

Here the second arrow is the multiplication map. Because M is an S module, it is annihilated by K , and so the composition of these two maps is 0. Thus the image of K/IK in $I \otimes_R M$ is in the kernel of the map $I \otimes_R M \rightarrow M$; by [Lemma 9](#), this kernel is isomorphic to $H_1(G, M)$. $\square$

We can now rewrite [Equation \(11\)](#), leveraging the previous lemmas:

$$\begin{array}{ccc}
 \text{Tor}_1^S(I \otimes_R S, M) & \xleftarrow{\quad} & H_2(G, M) & \text{Lemma 10} \\
 \downarrow & & \downarrow & \\
 \text{Tor}_1^S(J, M) & \xleftarrow{\quad} & H_2(H, M) & \text{Lemma 9} \\
 \downarrow & & \downarrow & \\
 K/IK \otimes_S M & \xleftarrow{\quad} & \overline{N} \otimes_S M & \text{Lemma 4, Lemma 11} \\
 \downarrow & \searrow & \downarrow & \\
 I \otimes_R S \otimes_S M & \xleftarrow{\quad} & H_1(G, M) & \text{Lemma 12, Lemma 13} \\
 \swarrow \sigma & \downarrow \pi & \downarrow & \\
 M & \xleftarrow{\tau} J \otimes_S M & \xleftarrow{\quad} & H_1(H, M) & \text{Lemma 9} \\
 & \downarrow & & \downarrow & \\
 & 0 & & 0 &
 \end{array}$$

There is just a bit to say about the induced map from $H_1(G, M)$ to $H_1(H, M)$: it works because π is surjective, so the kernel of σ maps onto the kernel of τ , which is $H_1(H, M)$ by [Lemma 9](#). We capture this new induced exact sequence as a theorem:

THEOREM 14 (Stallings): *For a quotient group $G/N \cong H$ and a $\mathbb{Z}[H]$ module M , there is an exact sequence*

$$H_2(G, M) \rightarrow H_2(H, M) \rightarrow \bar{N} \otimes M \rightarrow H_1(G, M) \rightarrow H_1(H, M) \rightarrow 0 \quad (12)$$

Applications

The most important case of [Theorem 14](#) is when $M = \mathbb{Z}$, in which case [Proposition 8](#) says that $\bar{N} \otimes M$ should be the biggest G/N invariant quotient module of $\bar{N}$. Since G acts on N via conjugation, this means that

$$\bar{N} \otimes M \cong N/[G, N]$$

We capture this case separately:

COROLLARY 15: *There is an exact sequence*

$$H_2(G, \mathbb{Z}) \rightarrow H_2(G/N, \mathbb{Z}) \rightarrow N/[G, N] \rightarrow G/[G, G] \rightarrow G/N[G, G] \rightarrow 0$$

Let's focus on the case of $F/R \cong H$ being a **presentation** of H ; that is, F is a free group. In particular, $H_2(F, \mathbb{Z}) = 0$, and so our exact sequence simplifies to

$$0 \rightarrow H_2(H, \mathbb{Z}) \rightarrow R/[F, R] \rightarrow F/[F, F] \rightarrow H/[H, H] \rightarrow 0 \quad (13)$$

The kernel of the map $R/[F, R]$ to $F/[F, F]$ is just those elements in $[F, F]$, and of course $[F, R] \leq [F, F]$, and thus we have proved

THEOREM 16 (Hopf): $H_2(H, \mathbb{Z}) \cong (R \cap [F, F])/[F, R]$.

Let's see what this theorem says about finite groups H . We will work in the group $F/[F, R]$, where $R/[F, R]$ is a central subgroup. Now $F/[F, F]$ is a torsion-free Abelian group, so any element of finite order in $R/[F, R]$ is in $[F, F]/[F, R]$ as well. Conversely, since H is finite, the center (containing $R/[F, R]$) is finite index (since $[F : R]$ is finite), and a theorem of Schur then implies that the commutator $[F, F]/[F, R]$ is finite. This shows that $(R \cap [F, F])/[F, R]$ is precisely the torsion subgroup of $R/[F, R]$. This is important because if F/R is a **finite presentation** of H , which means F is finitely generated and R is finitely normally generated, then $F/[F, R]$ has a finite presentation. Then a theorem of Nielsen-Schreier gives an algorithm to find a presentation for the finite index subgroup $R/[F, R]$. Finally, computing the Abelianization of a finitely presented group is easy algorithmically, and then taking the torsion subgroup gives us $H_2(H, \mathbb{Z})$. All this is to say that there is an efficient algorithm to compute this homology group for finite groups H .

For the next results, we need some (non-standard) terminology. Throughout the rest of this section, all groups will be finitely presented. For a group G , we define $\gamma(G)$ to

be the minimal number of generators necessary to generate G . For a given presentation

$$G = \langle g_1, \dots, g_n \mid r_1, \dots, r_m \rangle \quad (14)$$

we define the **deficiency** as $n - m$. Finally, we define the deficiency $\delta(G)$ of G as the maximum deficiency across all presentations of G . To see this is well-defined, note that the fact G has a finite presentation immediately puts a lower bound on $\delta(G)$. In the other direction, the Abelianization of the presentation above has rank at least $n - m$, and so the rank of $G/[G, G]$ is an upper bound on $\delta(G)$. In fact, we can use Equation (13) to give an even stricter bound.

THEOREM 17 (Hall): *For all groups G , we have*

$$\delta(G) \leq \text{rk}(H_1(G, \mathbb{Z})) - \gamma(H_2(G, \mathbb{Z}))$$

Proof. Consider a presentation like Equation (14). In this case, $G \cong F/R$, where $\gamma(F) = n$ and R can be normally generated by m elements. Note that this means that $\gamma(R/[F, R]) \leq m$. Only generators of $R/[F, R]$ that end up as a non-trivial element in $F/[F, F]$ can lead to a potential reduction in rank in the quotient $G/[G, G]$. If we call these number of generators r , then Equation (13) implies that

$$\begin{aligned} r &\geq \gamma(F) - \text{rk}(G/[G, G]) \\ &= n - \text{rk}(H_1(G, \mathbb{Z})) \end{aligned}$$

How many of the $\gamma(R/[F, R])$ generators are non-trivial in $F/[F, F]$? Well since $R/[F, R]$ is a finitely generated abelian group, we can choose a generating set for each cyclic factor group. Then, in this $\mathbb{Z}$ -basis, every generator of $H_2(G, \mathbb{Z})$ leads to a finite cyclic quotient, which must end up trivial in the torsion-free group $F/[F, F]$. If we let the number of generators for the subgroup $H_2(G, \mathbb{Z})$ be k , then we have

$$\begin{aligned} m - \gamma(H_2(G, \mathbb{Z})) &\geq \gamma(R/[F, R]) - k \\ &\geq r \end{aligned}$$

Combined with the previous inequality for r , we get our desired result. $\square$

It might seem like we used quite a few crude estimates in the proof of Hall's theorem, and thus better bounds should be realizable. However, in full generality, this bound is tight. There are **superperfect** groups, whose first and second homology is trivial, and thus by Hall's inequality must have non-positive deficiency. But the smallest superperfect group ($SL(2, 5)$) has deficiency zero, realized by the presentation

$$SL(2, 5) \cong \langle a, b \mid a^2 = b^3 = (ab)^5 \rangle$$

It is an open question if every superperfect group has deficiency zero.

Here's another case of Hall's bound being tight. Since the rank of $H_1(G, \mathbb{Z})$ is certainly bounded by $\gamma(G)$, Hall's theorem implies that $\delta(G) \leq \gamma(G)$. What about groups where equality happens? Surely a free group, which can be given a presentation with no relators, satisfies equality. The following theorem shows those are the only ones.

THEOREM 18 (Magnus): *If $\delta(G) = \gamma(G)$ for a finitely presented group G , then G is free.*

Proof. Our proof follows the argument in [Sta67]. Let $G \cong F/R$ be a presentation realizing the deficiency $\delta(G)$, with $\gamma(F) = n$ and R normally generated by $r = n - \gamma(G)$ relators. We write K for the Abelian group $R/[F, R]$. Note that Equation (13) can be written as

$$0 \rightarrow H_2(G, \mathbb{Z}) \rightarrow K \rightarrow \mathbb{Z}^n \rightarrow \overline{G} \rightarrow 0$$

Because $\overline{G}$ has rank at most $\gamma(G)$, we see that $\text{rk}(K) \geq r$. Since K is generated by at most r elements, we then have that $K \cong \mathbb{Z}^r$. Thus the map $K \rightarrow \mathbb{Z}^n$ is injective, and we see that $H_2(G, \mathbb{Z}) = 0$.

Then $\text{rk}(\overline{G}) = n - r = \gamma(G)$ and thus $\overline{G} \cong \mathbb{Z}^{\gamma(G)}$. Now let $\hat{F}/\hat{R} \cong G$ be a presentation with $\gamma(G)$ generators. Then Equation (13) together with our computations of the homology of G give

$$0 \rightarrow \hat{R}/[\hat{F}, \hat{R}] \rightarrow \mathbb{Z}^{\gamma(G)} \rightarrow \mathbb{Z}^{\gamma(G)} \rightarrow 0$$

The map on the right, being surjective, must also be injective. Thus $\hat{R}/[\hat{F}, \hat{R}] = 0$. In other words, $\hat{R} = [\hat{F}, \hat{R}] = [\hat{R}, \hat{F}]$. Inductively this gives

$$\begin{aligned} \hat{R} &= [\hat{R}, \hat{F}] \\ &= [\hat{R}, \hat{F}, \hat{F}] \\ &= \dots \end{aligned}$$

So $\hat{R}$ is in every term of the lower central series for $\hat{F}$. But free groups are residually nilpotent, and thus we must have $\hat{R}$ is trivial. Thus $G \cong \hat{F}$ is free of rank $\gamma(G)$. $\square$

For completeness, we include a proof that free groups are residually nilpotent, based on the argument in [KM79]. Note that being residually nilpotent implies that for any element $g \neq 1$ in G , there's a homomorphism $\phi : G \rightarrow H$, where H is nilpotent and $\phi(g) \neq 1$. This property obviously passes to subgroups, by restricting the domain of the homomorphisms. Now let F be a free group, and $g \in F$ a non-trivial element. Then the word representing g only uses finitely many of the basis elements for F , and thus $g \in \hat{F}$, a finitely-generated free subgroup of F . Any homomorphism $\phi : \hat{F} \rightarrow H$ clearly extends to F by sending all other basis elements to the identity. Thus, to show free groups are residually nilpotent, it suffices to focus on the finitely generated ones.

We can simplify even further by noting that a free group of rank n is contained as a subgroup of index $n - 1$ in F_2 , the free group on 2 generators. This follows from either the Nielsen-Schreier subgroup theorems, or covering space theory applied to a figure-eight graph. Since

residual nilpotency passes to subgroups, we can then focus all of our attention on showing F_2 is residually nilpotent.

We do that by first showing F_2 is linear. Define the matrices $A, B \in SL(2, \mathbb{Z})$ as

$$\begin{aligned} A &= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} & B &= \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \\ A^n &= \begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix} & B^n &= \begin{pmatrix} 1 & 0 \\ 2n & 1 \end{pmatrix} \end{aligned}$$

Thus both A and B generate infinite cyclic subgroups of $SL(2, \mathbb{Z})$. Consider the subsets $X, Y \subset \mathbb{Z}^2$ defined by

$$\begin{aligned} X &= \{(x, y) \in \mathbb{Z}^2 \mid |x| > |y|\} \\ Y &= \{(x, y) \in \mathbb{Z}^2 \mid |y| > |x|\} \end{aligned}$$

Note that for $v = (x, y) \in Y$, we have $A^m v = (x + 2my, y)$. But when $m \neq 0$,

$$\begin{aligned} |x + 2my| &\geq 2|m||y| - |x| \\ &> (2|m| - 1)|y| \\ &\geq |y| \end{aligned} \quad (v \in Y)$$

So $A^m v \in X$. Similarly, $B^n u \in Y$ for $u \in X$. The ping-pong lemma then shows $\langle A, B \rangle$ is a free group of rank 2, and we thus get a linear representation of F_2 . This representation ends up in the subgroup $\Gamma(1) \leq SL(2, \mathbb{Z})$, defined as

$$\Gamma(k) = \{C \in SL(2, \mathbb{Z}) \mid C \equiv I \pmod{2^k}\}$$

Here the equivalence is on each matrix entry. Said another way, every $C \in \Gamma(k)$ can be written as $C = I + D$, where every entry in D is divisible by 2^k . The $\Gamma(k)$ are normal subgroups, since they are the kernels of the natural reduction map $SL(2, \mathbb{Z}) \rightarrow SL(2, \mathbb{Z}/2^k\mathbb{Z})$. Note that if $C = I + D \in \Gamma(k)$, then we have

$$C^2 = (I + D)^2 = I + 2D + D^2$$

And all entries of both $2D$ and D^2 are divisible by 2^{k+1} . Thus $C^2 \in \Gamma(k+1) \subset \Gamma(k)$. This shows that $\Gamma(k)/\Gamma(k+1)$ contains only elements of order dividing 2; because $\Gamma(k+1)$ is finite index in $SL(2, \mathbb{Z})$ for each $k \geq 0$, it follows that $\Gamma(k)/\Gamma(k+1)$ is a finite elementary Abelian 2-group.

Finally, $\bigcap_{k \geq 1} \Gamma(k) = \{1\}$, since only 0 can be divisible by all powers of 2. So for any nontrivial element $g \in \Gamma(1)$, there is a k with $g \notin \Gamma(k)$. Then $\Gamma(1)/\Gamma(k)$, as a composition of finite 2-groups, is a finite 2-group, and thus nilpotent. The quotient map $\Gamma(1) \rightarrow \Gamma(1)/\Gamma(k)$ does not contain g in the kernel. So $\Gamma(1)$ is residually nilpotent, and thus so is the subgroup $\langle A, B \rangle \cong F_2$. Note that this proof also shows F_2 (and hence all free groups) are residually finite.

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