

Let's talk about classifying fiber bundles over spheres. Specifically, let F be our fiber, G the structure group, and S^n the sphere. Then [Ste99, §18] shows the fiber bundles are classified by the orbits of $\pi_0(G, e)$ acting on $\pi_{n-1}(G, e)$ via conjugation (here $\pi_0(G, e) = G/I$ where I is the identity component of G).

In the case $F = \mathbb{R}^m$ and $G = \text{GL}(m, \mathbb{R})$, then $\pi_0(G, e)$ has order 2, generated by the identity matrix and the matrix

$$M = \begin{pmatrix} -1 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

Let's focus on the case $n = m = 2$. Now *oriented* 2-plane bundles over any manifold N are classified by $H^2(N, \mathbb{Z})$, and the group structure is given by tensor product. So for S^2 , we expect infinitely many of them, indexed by $\mathbb{Z}$. By thinking of S^2 as $\mathbb{CP}^1$, these can be written as $\mathcal{O}(n)$, which are complex line bundles, and thus oriented 2-dimensional real bundles. $\mathcal{O}(-1)$ is a generator, known as the tautological line bundle, with $\mathcal{O}(-n) = \bigoplus_{i=1}^n \mathcal{O}(-1)$. $\mathcal{O}(1)$ is the other generator, which is the dual of the tautological line bundle.

$\pi_1(\text{GL}(2, \mathbb{R}), e) \cong \mathbb{Z}$ as can be seen by using the Gram-Schmidt process to deformation retract to $O(2)$. So one might expect these complex line bundles give all the real 2-plane bundles over S^2 . But that's wrong, because M acts non-trivially on $\pi_1(\text{GL}(2, \mathbb{R}), e)$. Specifically, every loop in $\pi_1(\text{GL}(2, \mathbb{R}), e)$ ends up in the identity component, which deformation retracts to $SO(2)$. So we can think of such a loop as a map from S^1 to $SO(2) \cong S^1$, and up to homotopy these are given by the loops $\gamma_k : \theta \mapsto k\theta$ with $k \in \mathbb{Z}$. The matrix representation for γ_k is

$$\Gamma_k(\theta) = \begin{pmatrix} \cos(k\theta) & -\sin(k\theta) \\ \sin(k\theta) & \cos(k\theta) \end{pmatrix}$$

Conjugating this by M gives

$$\begin{aligned} M^{-1}\Gamma_k M(\theta) &= \begin{pmatrix} \cos(k\theta) & \sin(k\theta) \\ -\sin(k\theta) & \cos(k\theta) \end{pmatrix} \\ &= \Gamma_{-k}(\theta) \end{aligned}$$

So when $k \neq 0$, the pair $\{\gamma_k, \gamma_{-k}\}$ form a π_0 orbit within π_1 . We would thus expect that two bundles with these clutching functions are isomorphic. We give an explicit proof of this below, when $k = 1$. This thus serves as a cautionary tale of trying to classify bundles over the sphere without taking into account the π_0 action!

The total space of $\mathcal{O}(-1)$ can be written as the subspace $E \subset \mathbb{CP}^1 \times \mathbb{C}^2$, where

$$E = \{([\ell]; v) \mid v \in \ell\}$$

with projection on the first factor as the bundle map π . Thinking of $[\ell] = [x : y]$, the trivialization charts are $U_x = \{[x : y] \mid x \neq 0\}$ and U_y (defined similarly). We have trivializations

$$\begin{aligned} U_x \times \mathbb{C} &\rightarrow \pi^{-1}(U_x) & U_y \times \mathbb{C} &\rightarrow \pi^{-1}(U_y) \\ ([x : y], z) &\mapsto ([x : y]; (z, \frac{y}{x}z)) & ([x : y], z) &\mapsto ([x : y]; (\frac{x}{y}z, z)) \end{aligned}$$

Thinking of the equator in $\mathbb{CP}^2$ as $\{[1 : \lambda] \mid |\lambda| = 1\}$, the clutching function is then

$$\begin{aligned} \pi^{-1}(U_x) &\rightarrow U_x \times \mathbb{C} & \rightarrow \pi^{-1}(U_y) \\ ([1 : \lambda]; v) &\mapsto ([1 : \lambda], v_1) \mapsto ([1 : \lambda]; \frac{1}{\lambda}v) \end{aligned}$$

Thus the clutching function is γ_{-1} , as expected for $\mathcal{O}(-1)$.

The total space for $\mathcal{O}(1)$ can be taken as $X = \mathbb{CP}^2 \setminus [0 : 0 : 1]$, with the bundle map π given by

$$\pi([x : y : z]) = [x : y]$$

Trivializations are given by

$$\begin{aligned} U_x \times \mathbb{C} &\rightarrow \pi^{-1}(U_x) & U_y \times \mathbb{C} &\rightarrow \pi^{-1}(U_y) \\ ([x : y], z) &\mapsto [x : y : xz] & ([x : y], z) &\mapsto [x : y : yz] \end{aligned}$$

Thinking again of the same equator as above, the clutching function is then

$$\begin{aligned} \pi^{-1}(U_x) &\rightarrow U_x \times \mathbb{C} & \rightarrow \pi^{-1}(U_y) \\ [1 : \lambda : z] &\mapsto ([1 : \lambda], z) \mapsto [1 : \lambda : \lambda z] \end{aligned}$$

Thus the clutching function is γ_1 , as expected for $\mathcal{O}(1)$.

To see a bundle isomorphism between E and X , we can start with a map that is the identity on the unit sphere bundle S^3 , and then scale appropriately. The unit sphere bundle in X is given by

$$\{[x : y : z] \mid |x|^2 + |y|^2 = |z|^2\}$$

and that suggests a map $H : X \rightarrow E$ given by

$$H([x : y : z]) = \left([x : y]; \frac{z(x, y)}{|x|^2 + |y|^2} \right)$$

However, this isn't well-defined; we can fix it with complex conjugation:

$$H([x : y : z]) = \left([x : y]; \frac{\bar{z}(x, y)}{|x|^2 + |y|^2} \right)$$

Note it's not surprising complex conjugation is involved; these bundles are isomorphic as *real* vector bundles, but not as complex ones, and complex conjugation is only $\mathbb{R}$ -linear. The inverse map can also be given explicitly:

$$H^{-1}([x, y]; v) = \begin{cases} [x : y : 0] & v = 0 \\ [v_1 : v_2 : |v|^2] & v \neq 0. \end{cases}$$

It's easy to see these are smooth maps, and routine to check they are inverses of each other. To see they're bundle isomorphisms, one can show what they do in each trivialization. For example, if $[x : y] \in U_x$ then H induces a map from $U_x \times \mathbb{C} \rightarrow U_x \times \mathbb{C}$ given by

$$([x : y], z) \mapsto \bar{z} \frac{|x|^2}{|x|^2 + |y|^2}$$

And in each fiber, up to scaling, this is complex conjugation, which is to be expected in an isomorphism between a bundle and its dual. Thus for $n > 0$

$$\mathcal{O}(n) \cong \oplus_{i=1}^n \mathcal{O}(1) \cong \oplus_{i=1}^n \mathcal{O}(-1) \cong \mathcal{O}(-n)$$

with the middle isomorphism only holding in the real vector bundle case as outlined above. Thus the 2-plane vector bundles over S^2 , which correspond to the orbits of $\pi_0(\mathrm{GL}(2, \mathbb{R}), e)$ on $\pi_1(\mathrm{GL}(2, \mathbb{R}), e)$, are given by $\mathcal{O}(n)$ for $n \geq 0$.

Bibliography

Steenrod, N. *The Topology of Fibre Bundles*. Princeton Landmarks in Mathematics and Physics. Princeton University Press, 1999. ISBN: 9780691005485. URL: https://books.google.com/books?id=m%5C_wrjoweDTgC.