

Introduction

Somewhat surprisingly, there exists compact manifolds that have the same homotopy and (co)homology groups, but they are not homotopy equivalent. The "smallest" example (in terms of dimension) comes from 3-dimensional Lens spaces; however, it takes a bit of work to set up when Lens spaces are homotopy equivalent and when they're not. Instead, we construct an example in four dimensions, which has the benefits of being easy to construct explicitly, and of having the two different manifolds be simply connected.

Setup

The key to these two 4-manifolds is that they are both fiber bundles over S^2 . In fact, they both are S^2 bundles over S^2 : $S^2 \hookrightarrow M \twoheadrightarrow S^2$.

This already shows M is a compact 4-manifold: cover S^2 by finitely many open sets $\{V_i\}$, such that $\overline{V}_i$ is compact and evenly covered. Then the preimage of $\overline{V}_i$ in M is homeomorphic to $\overline{V}_i \times S^2$, hence compact. Since M is the union of these finitely many preimages, it is also compact. It is a 4-manifold since it locally looks like $S^2 \times S^2$.

We make the further assumption now that the fiber bundle has a **section** $\sigma : S^2 \rightarrow M$. If we look at this at the level of homotopy, we get the following in the long exact sequence:

$$\cdots \pi_3(S^2) \rightarrow \pi_3(M) \xrightleftharpoons[\sigma_*]{} \pi_3(S^2) \xrightarrow{\tau_*} \pi_2(S^2) \rightarrow \pi_2(M) \xrightleftharpoons[\sigma_*]{} \pi_2(S^2) \xrightarrow{\tau_*} 0$$

The existence of σ shows that at the level of homotopy groups, the map $M \twoheadrightarrow S^2$ is surjective, and hence the τ_* maps are the zero maps by exactness. So this long exact sequence actually breaks up into short exact sequences

$$0 \rightarrow \pi_k(S^2) \rightarrow \pi_k(M) \xrightleftharpoons[\sigma_*]{} \pi_k(S^2) \rightarrow 0$$

In the case $k \in \{0, 1\}$ we get that $\pi_k(M) = 0$ since it's bookended by trivial groups in the exact sequence; thus M is connected and simply connected. In the other cases, all the groups are abelian, and the section σ_* shows these sequences split: $\pi_k(M) \cong \pi_k(S^2) \oplus \pi_k(S^2)$.

Since M is simply connected, it is orientable (otherwise an oriented double cover would give a subgroup of index 2 in $\pi_1(M)$). Then the Hurewicz theorem, universal coefficient theorem, and Poincaré duality give us the homology/cohomology of M :

$$H^n(M) \cong H_n(M) = \begin{cases} \mathbb{Z} & n = 0, 4 \\ \mathbb{Z}^2 & n = 2 \\ 0 & \text{otherwise} \end{cases}$$

Thus, if we have a fiber bundle $S^2 \hookrightarrow M \twoheadrightarrow S^2$ that also has a section, then the homotopy and (co)homology groups of M are uniquely determined.

The two manifolds

One of our fiber bundles with section is the trivial one: $S^2 \times S^2$. For the other nontrivial one, we'll think of S^2 as the complex projective line $\mathbb{CP}^1$. We *almost* have a fiber bundle $\pi : \mathbb{CP}^2 \rightarrow \mathbb{CP}^1$, given by just dropping the last coordinate: $[x : y : z] \rightarrow [x : y]$. However, this doesn't make sense at the single point $[0 : 0 : 1]$. In fact, the map $\pi : \mathbb{CP}^2 \setminus \{[0 : 0 : 1]\} \rightarrow \mathbb{CP}^1$ is a fiber bundle, with fiber $\mathbb{C}$. It's nicer if everything is compact though, so below we'll actually remove a whole neighborhood around $[0 : 0 : 1]$.

What we'll do to get the right manifold is two steps. First, we'll cut down the fiber of the above map so that we have a disk bundle over S^2 . Then we'll glue two copies of these total spaces together, and get a manifold that fits into $S^2 \hookrightarrow M \twoheadrightarrow S^2$.

To cut down the fiber, we'll remove the open set

$$U = \{[x : y : z] \in \mathbb{CP}^2 \mid \|z\|^2 > \|x\|^2 + \|y\|^2\}$$

Note that U is diffeomorphic to an open ball $\mathbb{B}^4$ via the map $[x : y : z] \rightarrow (\frac{x}{z}, \frac{y}{z})$ (the inverse map is $(x, y) \rightarrow [x : y : 1]$). This diffeomorphism extends to $\partial U \cong \partial \mathbb{B}^4 \cong S^3$. Now when we look at a fiber we get

$$\pi^{-1}([a : b]) \cap (\mathbb{CP}^2 \setminus U) = \{[a : b : z] \mid \|z\|^2 \leq \|a\|^2 + \|b\|^2\}$$

which is a disk (here a and b are constants), whose boundary circle lies in ∂U . In fact, the map $\pi : \partial U \rightarrow S^2$ is just the Hopf fibration.

So we have our new space and fiber bundle, $D^2 \hookrightarrow \mathbb{CP}^2 \setminus U \twoheadrightarrow S^2$. We will now glue two copies of this manifold together, by identifying the circles ∂D^2 in each copy; since gluing two copies of a disk this way gives a 2-sphere, we'll get a bundle whose fiber is S^2 , and we will finally have our second 4-manifold. Because we showed that U was a regular coordinate ball above, the manifold we get from our gluing is essentially $\mathbb{CP}^2 \# \mathbb{CP}^2$. However, when taking the connected sum of two oriented manifolds, the convention is that the gluing map on the sphere boundary is *orientation-reversing*, because this ensures the resulting space is also oriented. Since we want to use the identity map to glue our two sphere boundaries ∂U , we'll have to **reverse** the orientation of one summand. Thus to follow the connected sum convention, our second sphere bundle is the space $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$. Note we if we didn't reverse orientation on the second summand, our gluing map would be orientation-reversing, and no such map on S^3 can preserve the fibers of the Hopf fibration, and thus the boundaries of our disk fibers won't line up to glue together for S^2 fibers. We'll see later on that $\mathbb{CP}^2 \# \mathbb{CP}^2$ cannot be a bundle in this way, no matter how we try and do the gluing.

Finally, we need a section $\sigma : \mathbb{CP}^1 \rightarrow \mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$. But that's easy: send $[x : y]$ to $[x : y : 0]$ in one of the summands.

Not the same

We thus have two closed, orientable 4-manifolds that have the same homotopy and (co)homology groups: $M_1 \cong S^2 \times S^2$ and $M_2 \cong \mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$. But these manifolds are not homotopy equivalent (and thus certainly not diffeomorphic). To see this, we'll use the cohomology rings $H^*(M_i)$.

The cohomology ring of M_1 is straightforward to compute: the cohomology of a product of manifolds is the tensor product of their cohomologies:

$$\begin{aligned} H^*(S^2 \times S^2) &\cong H^*(S^2) \otimes H^*(S^2) \\ &\cong \mathbb{Z}[\alpha, \beta] / \langle \alpha^2, \beta^2 \rangle \end{aligned}$$

The two important things to note about this structure are:

P1 The generator for $H^4(M_1)$ is a product of two elements from $H^2(M_1)$.

P2 Every element of $H^2(M_1)$ squares to a non-generator (because of the presence of 2):

$$(n\alpha + m\beta)^2 = 2nm\alpha\beta$$

To look at the cohomology structure of M_2 , It helps to first look at the CW-structure of one of the summands $\mathbb{CP}^2 \setminus U$. As noted above, $\partial U \cong S^3$ and the restriction $\pi : \partial U \rightarrow \mathbb{CP}^1$ is just the Hopf fibration. In fact, $\mathbb{CP}^2 \setminus U$ is just the mapping cylinder of the Hopf fibration! To see this, consider the map $\rho : S^3 \times [0, 1] \rightarrow \mathbb{CP}^2 \setminus U$, given by

$$\rho(a, b, t) = [a : b : t]$$

This makes sense, since $t \leq \|a\|^2 + \|b\|^2 = 1$. To see this map is surjective, note that we have

$$\begin{aligned} \rho \left(\frac{x}{\sqrt{\|x\|^2 + \|y\|^2}}, \frac{y}{\sqrt{\|x\|^2 + \|y\|^2}}, 0 \right) &= [x : y : 0] \quad (z = 0) \\ \rho \left(\frac{x\bar{z}}{\|z\| \sqrt{\|x\|^2 + \|y\|^2}}, \frac{y\bar{z}}{\|z\| \sqrt{\|x\|^2 + \|y\|^2}}, \frac{\|z\|}{\sqrt{\|x\|^2 + \|y\|^2}} \right) &= [x : y : z] \quad (z \neq 0) \end{aligned}$$

The third coordinate in the second line above is in $[0, 1]$ because $\|z\|^2 \leq \|x\|^2 + \|y\|^2$. Also the coordinates in the second line above are homogeneous (multiplying x, y, z by a complex scalar does not change the coordinates), so we have a partial inverse where $z \neq 0$. Everything is compact, so ρ is a quotient map. The only possible identifications are when

$z = 0$, and then $\rho(a, b, 0) = \rho(\lambda a, \lambda b, 0)$ for any $\lambda \in S^1$ (we have to restrict to $\|\lambda\| = 1$ because we need (a, b) to stay on S^3). This is just the Hopf map in $S^3 \times \{0\}$. Since these are the same identifications for the mapping cylinder of the Hopf map, the uniqueness of quotients shows $\mathbb{CP}^2 \setminus U$ is homeomorphic to the mapping cylinder.

The gluing map for the connected sum $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ is the identity map, and so we are gluing the boundaries of two mapping cylinders by the identity. Thus M_2 can be built by taking two disjoint S^2 , and then gluing on an $S^3 \times [0, 1]$ by the Hopf map at each end. [In this description, $S^3 \times \{\frac{1}{2}\}$ is where we glue the two cylinders.] Although it's not really important, the two S^2 are the image of the section $\sigma(S^2)$ and the corresponding image in the other summand (let's call it $\tilde{\sigma}(S^2)$). Since $S^3 \times [0, 1]$ is the quotient of $D^3 \times [0, 1]$, we see that the CW-structure of M_2 has no 3-cells and two 2-cells, and thus $H_2(M_2)$ is generated by the classes of $\sigma(S^2)$ and $\tilde{\sigma}(S^2)$. Via duality, we can then generate $H^2(M_2)$ by the duals of these homology classes, which we'll call γ and δ .

Since these cohomology classes are represented by submanifolds, we can compute the cup product by: taking the submanifold representatives; perturbing them to intersect transversally; taking the dual of the intersection. Since $\sigma(S^2)$ and $\tilde{\sigma}(S^2)$ are disjoint, we see that $\gamma \smile \delta = 0$. Meanwhile, we can see that $\gamma^2 \neq 0$ similarly to how this works in $\mathbb{CP}^2$. Namely, the isotopy $f_t : \mathbb{CP}^1 \rightarrow \mathbb{CP}^2$ given by

$$f_t([a : b]) = [a \cos(\frac{\pi t}{2}) : b : a \sin(\frac{\pi t}{2})]$$

has $f_0(S^2) = \sigma(S^2)$ and allows us to compute $f_0(S^2) \cap f_1(S^2) = [0 : 1 : 0]$, a single point of (transverse) intersection, and thus we expect γ^2 to be a generator. Now $f_1(S^2)$ does not lie in $\mathbb{CP}^2 \setminus U$, but $f_{\frac{1}{2}}(S^2)$ does (since $\tan(\frac{\pi t}{2}) \leq 1$ for $t \leq \frac{1}{2}$), and the intersection of $f_0(S^2)$ and $f_{\frac{1}{2}}(S^2)$ is still the single point $[0 : 1 : 0]$. Thus γ^2 is a generator of $H^4(\mathbb{CP}^2 \# \overline{\mathbb{CP}^2})$. Equivalent reasoning shows δ^2 is also a generator, and the orientation reversal between the two summands $\mathbb{CP}^2$ and $\overline{\mathbb{CP}^2}$ means $\gamma^2 = -\delta^2$. Since having an element of H^2 square to a generator of H^4 is in opposition to **P2**, we see that $H^*(M_1) \not\cong H^*(M_2)$ as rings, so these manifolds are not homotopic (nor diffeomorphic).

Cohomology of connected sums

In this section we'll show how to generally obtain the cohomology ring of a connected sum of manifolds, and then use this in the specific case of M_1 and M_2 above. Note we've already shown M_1 and M_2 are not homotopic; what we do here is just to put things in a broader context.

Let M and N be compact oriented manifolds of dimension $n > 1$. Then

$$\begin{aligned}\tilde{H}^*(M \times N) &\cong \tilde{H}^*(M) \otimes \tilde{H}^*(N) \\ \tilde{H}^*(M \vee N) &\cong \tilde{H}^*(M) \oplus \tilde{H}^*(N)\end{aligned}$$

via categorical nonsense. We use reduced cohomology here to avoid the tedious bookkeeping of the $\oplus$ operation in degree 0. Now consider the connected sum $M\#N$, defined as gluing $\hat{M} = M \setminus U$ and $\hat{N} = N \setminus V$ (with U and V regular coordinate balls) along $\partial U \cong \partial V \cong S^{n-1}$. Let $i : S^{n-1} \hookrightarrow M\#N$ be the embedding of the boundary sphere where the connected sum gluing takes place. Note that if we collapse the S^{n-1} embedded submanifold, we get $M \vee N$. The key fact here is that

$$S^{n-1} \xrightarrow{i} M\#N \xrightarrow{p} M \vee N$$

is a **cofibration**. To see this, note that S^{n-1} has a collar neighborhood in each of the M and N summands (post regular ball removal). These collar neighborhoods glue together in $M\#N$ to make an open neighborhood of S^{n-1} that deformation retracts to S^{n-1} . Then together with some bump functions, by [Bre93, VII.1.5] the inclusion map is a cofibration, with quotient space $M \vee N$.

This is a key fact because [Bre93, VII.1.7] together with the long exact sequence of a pair for cohomology gives the long exact sequence

$$\dots \tilde{H}^k(M \vee N) \xrightarrow{p^{*k}} \tilde{H}^k(M\#N) \xrightarrow{i^{*k}} \tilde{H}^k(S^{n-1}) \xrightarrow{\delta^{*k}} \tilde{H}^{k+1}(M \vee N) \dots$$

Consider the ring homomorphism $p^* : H^*(M \vee N) \rightarrow H^*(M\#N)$. We will show this is surjective and compute the kernel. Now $\tilde{H}^k(S^{n-1}) \neq 0$ only for $k = n - 1$, thus when $k \notin \{n - 1, n\}$ p^* is surjective with no kernel in degree k . So we can focus attention on the subsequence

$$0 \rightarrow \tilde{H}^{n-1}(M \vee N) \xrightarrow{p^{*(n-1)}} \tilde{H}^{n-1}(M\#N) \xrightarrow{i^{*(n-1)}} \tilde{H}^{n-1}(S^{n-1}) \xrightarrow{\delta^{*(n-1)}} \tilde{H}^n(M \vee N) \xrightarrow{p^{*n}} \tilde{H}^n(M\#N) \rightarrow 0$$

At the level of groups, this looks like

$$\dots \xrightarrow{i^{*(n-1)}} \mathbb{Z} \xrightarrow{\delta^{*(n-1)}} \mathbb{Z} \oplus \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow 0$$

which shows $\delta^{*(n-1)}$ is injective, and then $i^{*(n-1)}$ is the zero map by exactness. Thus in degree $n - 1$ we also have p^* surjective with trivial kernel. Then we have a ring isomorphism

$$\tilde{H}^*(M\#N) \cong \tilde{H}^*(M \vee N) / \text{Im}(\delta^{*n})$$

So what is $\text{Im}(\delta^{*n})$? We can compute it directly based on what δ^* does; note that every manifold in sight is compact and oriented, so has a preferred **orientation form**, a generator of its top cohomology. Let $[\sigma] \in \tilde{H}^{n-1}(S^{n-1})$ be that generator, with $\sigma \in \text{Hom}(C_{n-1}(S^{n-1}), \mathbb{Z})$,

so that $\sigma([S^{n-1}]) = 1$. The first step of δ^* is to build $\tau \in \text{Hom}(C_{n-1}(M\#N), \mathbb{Z})$ where $\tau(i^*[S^{n-1}]) = 1$ but is zero on all other $n-1$ -dimensional subcomplexes. We then build $\partial\tau \in \text{Hom}(C_n(M\#N), \mathbb{Z})$ such that $\partial\tau(\kappa) = \tau(\partial\kappa)$. Finally, we build $\nu \in \text{Hom}(C_n(M \vee N), \mathbb{Z})$ such that for all $\kappa \in C_n(M\#N)$, $\nu(p_*(\kappa)) = \partial\tau(\kappa)$. The upshot of all this is that $\delta^*[\sigma] = [\nu]$.

To make the calculation of $\text{Im}(\delta^{*n})$ more palatable, let j_M and j_N be the inclusions of M and N into $M \vee N$, and note that $j_M^* + j_N^* : \tilde{H}^*(M \vee N) \rightarrow \tilde{H}^*(M) \oplus \tilde{H}^*(N)$ is an isomorphism. Recalling that $\hat{M}$ is the "M part" of $M\#N$, we see that

$$\begin{aligned}\tau(\partial\hat{M}) &= 1 \\ \partial\tau(\hat{M}) &= 1 \\ j_M^*[\nu]([M]) &= [\nu]([j_M(M)]) \\ &= [\nu](p_*[\hat{M}]) \\ &= 1\end{aligned}$$

The key here is that the image of $\hat{M}$ in $M \vee N$ is M . Thus $j_M^*[\nu] = j_M^*\delta^*[\sigma]$ is the orientation form $[\mu_M] \in \tilde{H}^n(M)$. A similar calculation works for N , except for the fact that $\tau(\partial\hat{N}) = -1$ because of the orientation reversal when gluing. We thus have that $(j_M^* + j_N^*)\text{Im}(\delta^{*n}) = (\mu_M, -\mu_N)$ and so

$$\tilde{H}^*(M\#N) \cong \frac{\tilde{H}^*(M) \oplus \tilde{H}^*(N)}{\langle \mu_M - \mu_N \rangle}$$

We can now look at the cohomology rings of the spaces we've discussed in this note. One calculation we won't do (but that can be found pretty much anywhere) is that

$$H^*(\mathbb{CP}^n) \cong \mathbb{Z}[\alpha] / \langle \alpha^{n+1} \rangle$$

We're going back to unreduced cohomology, so the first isomorphisms for S and T below are missing a quotient in degree 0, but the end result is correct and hopefully the meaning is clear:

- $R = H^*(S^2 \times S^2) \cong H^*(\mathbb{CP}^1) \otimes H^*(\mathbb{CP}^1) \cong \mathbb{Z}[\alpha, \beta] / \langle \alpha^2, \beta^2 \rangle$
- $S = H^*(\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}) \cong \frac{H^*(\mathbb{CP}^2) \oplus H^*(\mathbb{CP}^2)}{\langle \mu_M + \mu_N \rangle} \cong \mathbb{Z}[\alpha, \beta] / \langle \alpha^3, \beta^3, \alpha\beta, \alpha^2 + \beta^2 \rangle$
- $T = H^*(\mathbb{CP}^2 \# \mathbb{CP}^2) \cong \frac{H^*(\mathbb{CP}^2) \oplus H^*(\mathbb{CP}^2)}{\langle \mu_M - \mu_N \rangle} \cong \mathbb{Z}[\alpha, \beta] / \langle \alpha^3, \beta^3, \alpha\beta, \alpha^2 - \beta^2 \rangle$

As noted before, in R we have $(k\alpha + l\beta)^2 = 2kl\alpha\beta$ is a non-generator. Meanwhile, α^2 in S is a generator, so R and S are different rings. However we still have $(\alpha + \beta)^2 = 0$ in S . But in T we have $(k\alpha + l\beta)^2 = (k^2 + l^2)\alpha^2$, and so no nonzero degree 2 element squares to 0 in T . Thus the three rings are pairwise non-isomorphic, and all three spaces are pairwise non-homotopic.

Bibliography

We can now explain why $\mathbb{CP}^2 \# \mathbb{CP}^2$ is not an S^2 bundle over S^2 . Every S^2 bundle over S^2 is simply connected (and hence orientable). It can be constructed via clutching functions by looking at homotopy classes of maps of the equator into the structure group of an oriented S^2 bundle. That is, the S^2 bundles over S^2 are in bijective correspondence with $\pi_1(SO(3))$, which is cyclic of order two. There are thus only two distinct bundles, and we've already shown $S^2 \times S^2$ and $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ are two such. Since $\mathbb{CP}^2 \# \mathbb{CP}^2$ is different from both these spaces, it can't be an S^2 bundle over S^2 . Note how different the similar-looking spaces $\mathbb{CP}^2 \# \mathbb{CP}^2$ and $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ are!

Bibliography

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