

Sphere Bundles

In this short note, we construct two 4-manifolds with the same homotopy/homology/cohomology groups, which are not homotopic/homeomorphic/diffeomorphic.

The idea is to construct two S^2 bundles over S^2 , that have a section. Assuming M is such a space, it is a compact 4-manifold. Since S^2 is simply connected, the long exact sequence on homotopy groups shows $\pi_1(M)$ is trivial, and thus M is orientable. The section splits the long exact sequence on higher (Abelian) homotopy groups to give split short exact sequences

$$0 \rightarrow \pi_k(S^2) \rightarrow \pi_k(M) \rightarrow \pi_k(S^2) \rightarrow 0 \quad (1)$$

So that for all $k \geq 1$, we have $\pi_k(M) \cong \pi_k(S^2) \oplus \pi_k(S^2)$. Since M is compact and orientable, we can compute (co)homology with the Hurewicz theorem, universal coefficient theorem, and Poincaré duality:

$$H^n(M, \mathbb{Z}) \cong H_n(M, \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & n \in \{0, 4\} \\ \mathbb{Z}^2 & n = 2 \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

Thus all the groups we care about for M are completely determined by being such a bundle with a section. One choice for M is obvious: $M_1 = S^2 \times S^2$. The other choice is a little less obvious.

Recall that $\mathbb{C}P^2$ is constructed from $\mathbb{C}P^1$ by attaching a 4-ball via the Hopf map. Removing a smaller ball from this attached 4-ball then leaves the mapping cylinder of the Hopf map, which is a disk bundle over $\mathbb{C}P^1 \cong S^2$, with boundary S^3 . Gluing two such bundles via the identity map on the S^3 boundaries has the effect of gluing two disks along their boundaries in each fiber, thus giving an S^2 bundle over S^2 , with a clear section (send S^2 to one of the $\mathbb{C}P^1$). We write this as $M_2 = \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$, because the convention for connected sums of oriented manifolds is to glue with an orientation-reversing homeomorphism, but we want an orientation-preserving homeomorphism on our boundaries (namely, the identity map).

So our two 4-manifolds are M_1 and M_2 . We now show they are not homotopic by showing they have different cohomology rings; for the rest of this note our cohomology is over $\mathbb{Z}$. The Künneth theorem shows that

$$H^*(M_1) \cong \mathbb{Z}[\alpha, \beta] / \langle \alpha^2, \beta^2 \rangle \quad |\alpha| = |\beta| = 2 \quad (3)$$

Note that $(m\alpha + n\beta)^2 = 2mn\alpha\beta$ is **not** a generator of $H^4(M_1)$ (because of the presence of the 2 factor). To show $H^*(M_1) \not\cong H^*(M_2)$, we will exhibit a degree 2 element in $H^2(M_2)$ that squares to a generator of $H^4(M_2)$.

We have a smooth map $\sigma : \mathbb{CP}^1 \times \mathbb{R} \rightarrow \mathbb{CP}^2$ given by

$$\sigma_t([a : b]) = [a \cos(t) : b : a \sin(t)] \quad (4)$$

For small enough t , this map will end up in the "punctured" $\mathbb{CP}^2$ used to construct the connected sum for M_2 . We write S_t for the image of σ_t . Then for small enough $t \neq 0$, we have that on the level of homology, $[S_0] = [S_t]$, and we also have $S_0 \cap S_t = [0 : 1 : 0]$. Then if $\gamma \in H^2(M_2)$ is the dual of $[S_0]$, we have that γ^2 is a generator for $H^4(M_2)$, since it is the dual of the (transverse) intersection $[S_0 \cap S_t]$, and the intersection only has one point, so under an isomorphism $H^4(M_2) \cong \mathbb{Z}$, we have γ^2 being sent to ± 1 . Since this doesn't happen in $H^*(M_1)$, we have shown M_1 and M_2 are non-homotopic 4-manifolds, even though they have the same homotopy and (co)homology groups.