

Let's prove that for a topological space (X, x) , the fundamental group $\pi_1(X, x)$ acts on the higher homotopy groups $\pi_n(X, x)$.

The idea is to think of an element of $\pi_n(X, x)$ as being represented by a map $F : (I^n, \partial I^n) \rightarrow (X, x)$, with I the unit interval. Note that the pair $(I^n, \partial I^n)$ has the homotopy extension property. Now let $f : (I, \partial I) \rightarrow (X, x)$ be a loop based at x . We define a map $J : I^n \times I \rightarrow X$ using the homotopy extension property. Namely, we define

$$\begin{aligned} J(v, 0) &= F(v) \\ J(\delta, t) &= f(t) \end{aligned}$$

Here $\delta \in \partial I^n$ is an arbitrary point. These functions agree on their overlap since $F(\partial I^n) = \{x\}$. So the homotopy extension property allows us to extend J to a map on all of $I^n \times I$. We then define $F \cdot f$ (the action of f on F) by

$$F \cdot f \equiv J(\cdot, 1)$$

It might seem like this action would be trivial, because f is a loop based at x . But it can do some weird things between $f(0)$ and $f(1)$. Of course, if f is the *constant* path at x , then we can define $J(v, t) = F(v)$, and so $F \cdot f \equiv F$. By stacking two $I^n \times I$ together, we easily see that $F \cdot (fg) \equiv F \cdot f \cdot g$, so this is a group action.

We still need to show this is well-defined. In particular, if we choose a different representative loop g instead of f , or a different representative G instead of F , we still end up with the same element of $\pi_n(X, x)$. Part of this is also showing that even though the extension of J to $I^n \times I$ is not unique, we still have $J(\cdot, 1)$ is unique up to homotopy. Proving all of this uses the homotopy extension property again.

In particular, suppose we already have J representing $F \cdot f$ and K representing $G \cdot g$, with $F \sim G$ via a homotopy H and $f \sim g$ via a homotopy h . Note this includes the case that $F = G$ and $f = g$, but J and K are different extensions. We then want to define a map $L : I^n \times I \times I \rightarrow X$. We do that by defining it on $I^n \times I \times \{0\}$, and also on $\partial(I^n \times I) \times I$, then applying the homotopy extension property. Specifically,

$$\begin{aligned} L(v, s, 0) &= H(v, s) \\ L(\delta, s, t) &= h(s, t) \\ L(v, 0, t) &= J(v, t) \\ L(v, 1, t) &= K(v, t) \end{aligned}$$

Some explanations are necessary. We are using the fact that

$$\partial(I^n \times I) = \partial I^n \times I \cup I^n \times \{0, 1\}$$

We also need to check all these definition match on the overlaps. Doing that methodically:

$$\begin{aligned}
L(\delta, s, 0) &= h(s, 0) \\
&= x \\
&= H(\delta, s) \\
L(v, 0, 0) &= H(v, 0) \\
&= F(v) \\
&= J(v, 0) \\
L(v, 1, 0) &= H(v, 1) \\
&= G(v) \\
&= K(v, 0) \\
L(\delta, 0, t) &= h(0, t) \\
&= f(t) \\
&= J(\delta, t) \\
L(\delta, 1, t) &= h(1, t) \\
&= g(t) \\
&= K(\delta, t)
\end{aligned}$$

So everything agrees on overlaps and via the homotopy extension property we get our map L . But then the map

$$\zeta_s(v) = L(v, s, 1)$$

is a homotopy such that

$$\begin{aligned}
\zeta_0(v) &\equiv J(v, 1) \equiv F \cdot f \\
\zeta_1(v) &\equiv K(v, 1) \equiv G \cdot g
\end{aligned}$$

And thus ζ is a homotopy between $F \cdot f$ (coming from J) and $G \cdot g$ (coming from K). They thus give the same element in $\pi_n(X, x)$, and our action via $\pi_1(X, x)$ is well-defined.