

# Quaternion Actions

This short note investigates quaternions acting on themselves.

Let  $\mathbb{H}$  be the quaternions, and let  $q \in \mathbb{H}^*$  be a non-zero quaternion. Then  $q$  acts on  $\mathbb{H}$  via conjugation:

$$x \cdot q = q^{-1}xq \quad (1)$$

This action preserves length, since

$$\|q^{-1}xq\| = \|q\|^{-1} \cdot \|x\| \cdot \|q\| = \|x\| \quad (2)$$

In particular, it is an orthogonal action. Now  $\mathbb{R} \subset \mathbb{H}$  is central, and this action by  $\mathbb{H}^*$  thus fixes  $\mathbb{R}$  pointwise. By orthogonality, this action also fixes the complementary subspace  $\langle i, j, k \rangle \subset \mathbb{H}$ . This complementary subspace is clearly isomorphic (as a real vector space) to  $\mathbb{R}^3$ . We restrict our action to the unit quaternions  $S^3 \subset \mathbb{H}^*$ . Then we have  $S^3$  acting orthogonally on  $\mathbb{R}^3$ , giving a Lie group homomorphism  $S^3 \rightarrow O(3)$ . The kernel of this action is

$$S^3 \cap Z(\mathbb{H}) = S^3 \cap \mathbb{R} = \{\pm 1\} \quad (3)$$

We thus have an injective immersion  $S^3/\{\pm 1\} \rightarrow O(3)$ . Since both manifolds have dimension 3, this is a submersion, and hence an open map. But  $S^3$  is compact, so this is also a closed map. Since  $S^3$  is connected, we see that the image of this injective immersion (e.g., embedding) is the identity component of  $O(3)$ . In other words, we have a Lie group isomorphism

$$\mathbb{RP}^3 \cong S^3/\{\pm 1\} \cong SO(3) \quad (4)$$

We can generalize this conjugation action. Let  $S^3 \times S^3$  act on  $\mathbb{H}$  via

$$x \cdot (p, q) = p^{-1}xq \quad (5)$$

Since  $\|p\| = \|q\| = 1$ , this is still a length-preserving (and hence orthogonal) action on  $\mathbb{H}$ , which is vector-space isomorphic to  $\mathbb{R}^4$ . We thus have a Lie group homomorphism  $S^3 \times S^3 \rightarrow O(4)$ . The kernel of this map is easily seen to be the center of the diagonal subgroup in  $S^3 \times S^3$ , which as above is seen to be  $\{\pm(1, 1)\}$ . Now  $S^3 \times S^3$  and  $O(4)$  are both 6-dimensional, so the same arguments as above give a Lie group isomorphism

$$(S^3 \times S^3)/\{\pm(1, 1)\} \cong SO(4) \quad (6)$$

Now  $\{\pm(1, 1)\}$  is index 2 in the (normal) subgroup  $\{(\pm 1, \pm 1)\} \leq S^3 \times S^3$ . Combined with what we showed above, we have a sequence of two-sheeted coverings

$$S^3 \times S^3 \rightarrow SO(4) \rightarrow SO(3) \times SO(3) \quad (7)$$