

## Introduction

Let's look at a few different (right) actions coming from the unit quaternions  $S^3 \subset \mathbb{H}$ . For each action, we'll look at the: orbits; stabilizers; fixed points; core/kernel; and automorphisms. The idea is to do this with as little computation as possible. In particular, we will only compute the inverse and conjugates of elements in  $S^3$ . We write elements of  $S^3$  as  $\alpha + \beta j$ , with  $\alpha, \beta \in \mathbb{C}$  and  $|\alpha|^2 + |\beta|^2 = 1$ . Then the inverse of that element is  $\bar{\alpha} - \beta j$  and we have

$$(\bar{\alpha} - \beta j)(z)(\alpha + \beta j) = (z|\alpha|^2 + \bar{z}|\beta|^2) + \bar{\alpha}\beta(z - \bar{z})j \quad (1)$$

For a unit quaternion  $\alpha + \beta j$ , we say it has  $i$ -part  $\alpha$  and  $j$ -part  $\beta$ .

## Right action

We look at the action of  $G_1 = S^3$  on  $X_1 = \mathbb{CP}^1$ , given by treating a point  $[z : w]$  as a quaternion  $z + wj$  and doing (right) multiplication. So we define the action as

$$[z : w] \cdot (\alpha + \beta j) = [I : J]$$

where  $I$  and  $J$  are the  $i$ -part and  $j$ -part of  $(z + wj)(\alpha + \beta j)$ . This is clearly well-defined, since scaling  $z$  and  $w$  by some nonzero complex number does the same thing to our quaternion product.

The point  $[1 : 0]$  then gets mapped to  $[\alpha : \beta]$ . Since any point in  $\mathbb{CP}^1$  can be scaled so that  $|z|^2 + |w|^2 = 1$ , we see that our action of  $G_1$  on  $X_1$  is transitive. The stabilizer of  $[1 : 0]$  is precisely those unit quaternions with  $\beta = 0$ , which just gives a copy of  $S^1$  in  $S^3$  which we call  $H$ . We then have that  $\mathbb{CP}^1$  is just the coset space  $H/G$ , and this is the Hopf fibration.

A point  $g \in G_1$  fixes the coset  $Hx$  precisely when  $g \in H^x$ , or equivalently when  $xgx^{-1} \in H$ . For an arbitrary  $z \in H$ , we can see when it belongs to  $H^x$  using Equation (1), with  $x = \alpha + \beta j$ . We want this conjugate to have zero  $j$ -part, so that  $\bar{\alpha}\beta(z - \bar{z}) = 0$ . This only happens in three cases:

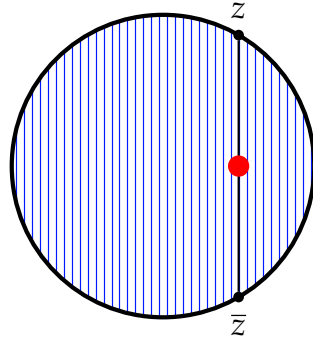
1.  $\beta = 0$ , and so  $x \in H$ .
2.  $\alpha = 0$ , and so  $x \in Hj$ .
3.  $z - \bar{z} = 0$ , and so  $z$  is real with  $z = \pm 1$ .

In the first two cases, it is independent of  $z \in H$ , and so we see that we have computed the normalizer of  $H$  in  $G_1$  as  $H \cup Hj$ . It also implies that for most points in  $H$ , they fix exactly two points, corresponding to the cosets  $H$  and  $Hj$ . In the third case, it is independent of  $x$ , and so we have computed the core of  $H$  to be  $\{\pm 1\}$ , which is the center of  $S^3$ .

Note that  $[1 : 0] \cdot j = [0 : 1]$ , so the elements of  $H$  fix the antipodal points  $[1 : 0]$  and  $[0 : 1]$ . To see that every element of  $G_1$  does something similar, we show that every element of  $G_1$  lies in a conjugate of  $H$ , which implies it fixes two points. To see this, [Equation \(1\)](#) shows the conjugates of elements in  $H$  look like

$$(z|\alpha|^2 + \bar{z}|\beta|^2) + \bar{\alpha}\beta(z - \bar{z})j$$

Let  $|\alpha|^2 = t$ , so that  $|\beta|^2 = 1 - t$ . Then treating  $S^1$  as the unit circle in the complex plane,  $z$  and  $\bar{z}$  are points on opposite sides of the  $x$ -axis, and  $tz + (1 - t)\bar{z}$  is a point on the line segment between them. Thus by choosing  $z$  and  $t$ , we can build a conjugate that has  $i$ -part any complex number of norm at most 1.



Once  $t$  and  $z$  are fixed, the  $j$ -part looks like

$$2|\alpha|e^{-i\theta}|\beta|e^{i\sigma}\Im(z)$$

Here  $\theta$  is the angle for  $\alpha$  and  $\sigma$  the angle for  $\beta$ . Note that we've already chosen  $z$  so  $\Im(z)$  is effectively constant. But ignoring  $\theta$ , we can choose  $\sigma$  to get any complex number with appropriate norm for the conjugate to still lie in  $S^3$ . So yes, every element in  $S^3$  lies in a conjugate of  $H$ . Note that the degree of freedom represented by  $\theta$  is precisely the circle that is  $H$ . The two distinct conjugates of  $H$  our arbitrary element ends up in comes from being able to choose  $z$  or  $\bar{z}$  when building the  $i$ -part.

The degree of freedom in  $\theta$  actually shows every element of  $G_1$  has a centralizer that is a conjugate of  $H$ ; this also follows from the fact that any element in  $H$  has  $H$  as centralizer, and the just-established fact that every element of  $G_1$  lies in a conjugate of  $H$ .

Finally, the  $G_1$ -automorphisms of  $H/G_1$  are always  $N_{G_1}(H)/H$ , and as established above this is represented by 1 and  $j$ . The non-identity automorphism is given by mapping  $Hx$  to  $Hjx = jHx$ . Letting  $j$  act on the left and doing the quaternionic multiplication gives

$$j \cdot [z : w] = [-\bar{w} : \bar{z}]$$

and this is just the antipodal map. Now if  $g \in H^x$  then surely  $g \in H^{jx}$  since  $j$  normalizes  $H$ , and thus the two fixed points of  $g$  are  $Hx$  and  $Hjx$ , the latter thus being the antipodal point of the former.

## Conjugation action

Let's now change to an action of  $G_2 = S^3$  on itself, via conjugation:  $x \cdot g = xg$ . This action is far from being transitive, and we will describe the orbits at the end of the section. We will instead restrict our attention for now to a particular subset of  $G_2$  that is closed under conjugation.

If  $p \in \mathbb{H}$ , then  $|pg| = |gp| = |p|$  for  $g \in S^3$  since  $|g| = 1$ . Since

$$\langle u, v \rangle = \frac{|u+v|^2 - |u-v|^2}{4}$$

multiplication by an element  $g \in S^3$  is an orthogonal transformation. If  $u, v \in G_2$  then we have

$$\langle u, v \rangle = 0 \iff \langle u^g, v^g \rangle = 0$$

If  $u \in Z(G_2)$ , then this shows the elements orthogonal to  $u$  are closed under conjugation because  $u = u^g$ . In particular, since  $Z(S^3) = S^3 \cap \mathbb{R}$ , we see that the set  $X_2 = S^3 \cap \mathbb{R}^\perp$  is closed under conjugation. It is not closed under multiplication of course.

$X_2$  topologically is just the equator of  $S^3$  that is perpendicular to the line through  $\pm 1$ . That is,  $X_2$  is diffeomorphic to  $S^2$ , which is itself diffeomorphic to  $\mathbb{CP}^1$ . How is this action related to the previous one? Well, the centralizer of  $i$  is  $H$ , and so the point stabilizer of  $i$  in this conjugation action of  $G_2$  is  $H$ . Thus the actions of  $S^3$  on  $\mathbb{CP}^1$  and the orbit of  $i$  under conjugation are equivalent, since transitive actions are determined by their stabilizers. Although it is not necessary, we can show the orbit of  $i$  is in fact all of  $X_2$  (so that the  $G_2$  action is transitive), either by direct computation or via the bijection  $\phi : X_2 \rightarrow X_1$  given by

$$\phi(\alpha + \beta j) = \begin{cases} [\bar{\beta} : 1 + i\alpha] & \alpha \neq i \\ [1 : 0] & \alpha = i \end{cases}$$

This map is equivariant in the sense that  $\phi(x \cdot g) = \phi(x) \cdot g$  for all  $g \in S^3$ . Since the action on  $X_1$  is transitive, the action on  $X_2$  is transitive as well.

So our conjugation action is equivalent to the action of the previous section. Since we proved the conjugation action is orthogonal, we get a map  $F : S^3 \rightarrow O(3)$  given by restricting the conjugation action to  $X_2$ .  $F$  is a Lie group homomorphism since the action is smooth. The kernel of this map is the core of the action, which is  $\{\pm 1\}$ . Since the kernel has dimension 0,  $F(S^3)$  has dimension 3, which is equal to the dimension of  $O(3)$ . Thus  $F(S^3)$  is an open subset of  $O(3)$ . But  $S^3$  is compact and connected, so  $F(S^3)$  is also closed and connected in  $O(3)$ . So it is a component of  $O(3)$ , and since  $F$  is a homomorphism it's the identity component of  $O(3)$ .  $F$  thus induces an isomorphism

$$S^3 / \{\pm 1\} \cong SO(3)$$

Note that multiplying by  $-1$  is the antipodal map on  $S^3$ , so we thus get

$$\mathbb{RP}^3 \cong S^3 / \{\pm 1\} \cong SO(3)$$

The two fixed points for an arbitrary  $g \in S^3$  that we found in the previous section form the axis for the rotation represented by  $F(g)$ . The  $G_2$ -automorphisms of  $X_2$  are precisely what they were with  $G_1$  and  $X_1$ : the only non-trivial one is the antipodal map, and that together with  $SO(3)$  gives the full  $O(3)$ .

We can now describe the full conjugation action of  $S^3$  on itself. Clearly, every element of  $Z(S^3)$  will form an orbit by itself; thus there are the two trivial orbits of 1 and  $-1$ . For any other element we can write it as  $r + tx$  for  $r \in \mathbb{R}$  and  $x \in X_2$  with  $t = \sqrt{1 - r^2}$ . Since  $\mathbb{R}$  is central in  $\mathbb{H}$ , we see that

$$(r + tx)^g = r + tx^g$$

So that for every  $r \in (-1, 1)$ , the subset of  $S^3$  of all elements with real part  $r$  forms a conjugation-invariant subset  $T_r$ , and the conjugation action on this subset is the same as it is on  $X_2$  (note  $T_0 = X_2$ ). Topologically,  $S^3$  minus the "poles"  $\pm 1$  is diffeomorphic to an uncountable nested union of  $S^2$ , and the conjugation action is preserving each of those  $S^2$ , acting as  $SO(3)$ .

## Product action

We can extend the conjugation action to an action of  $G_3 = S^3 \times S^3$  on  $X_3 = S^3$  via

$$x \cdot (g, h) = g^{-1}xh$$

The conjugation action was just the restriction of this action to the diagonal subgroup  $D$  in  $G_3$ .  $G_3$  clearly acts transitively on  $X_3$ , since  $1 \cdot (1, h) = h$  for arbitrary  $h$ . The stabilizer of the point  $1 \in X_3$  is precisely  $D$ . The core of  $D$ , which is the set of elements that fix every point of  $X_3$ , is exactly  $\{\pm(1, 1)\}$ . Note that this isn't the same thing as the center of  $G_3$ , which is  $\{(\pm 1, \pm 1)\}$  and contains this core as a subgroup of index 2.

When does  $(g, h)$  have fixed points? Precisely when  $g^{-1}xh = x$ , which means  $g^x = h$ , and thus  $g$  and  $h$  are conjugate. Once we know they're conjugate, they're conjugate by exactly the elements of the coset  $C_{G_3}(g)x$ , and we showed earlier this is precisely  $H^y x$  for some  $y$ . Thus when  $g$  and  $h$  are conjugate,  $(g, h)$  fixes a circle in  $X_3$ . But when they aren't conjugate (like  $(g, 1)$  for  $g \neq 1$ ), then  $(g, h)$  has no fixed points.

If  $D^{(g, h)} = D$ , then it must be that  $g^{-1}h$  acts trivially on every  $x \in S^3$ ; that is,  $g^{-1}h \in Z(S^3)$ . This means that  $N_{G_3}(D)/D$  is represented by  $(1, 1)$  and  $(-1, 1)$ . The latter point here acts by negation, and so is the antipodal map on  $X_3$ .

Finally, as before, multiplication by elements of  $S^3$  is an orthogonal transformation, so this action gives us a Lie group homomorphism  $H : G_3 \rightarrow O(4)$ . Just as in the previous section, we get a Lie group isomorphism

$$(S^3 \times S^3) / \pm(1, 1) \cong SO(4)$$

The image of  $D$  in  $SO(4)$  is precisely the  $SO(3)$  from the previous section, fixing the subspace spanned by  $\pm 1 \subset S^3$ .

## Applications

We present two applications of the preceding results. First, note that a covering map of Lie groups always has kernel in the center. So we have shown that  $S^3 \times S^3$  covers only the Lie groups:

- $S^3 \times S^3$ .
- $S^3 \times SO(3)$ .
- $SO(4)$ .
- $SO(3) \times SO(3)$ .

We can put a few of these together, with the kernel of the action on  $\mathbb{CP}^1$ , to get

$$\begin{array}{ccccc}
 S^1 \times S^1 & \longrightarrow & S^3 \times S^3 & \twoheadrightarrow & S^2 \times S^2 \\
 \downarrow & & \downarrow / \pm(1,1) & & \\
 SO(2) \times SO(2) & \longrightarrow & SO(4) & \twoheadrightarrow & S^2 \times S^2 \\
 \downarrow & & \downarrow / \pm I & & \\
 SO(2) \times SO(2) & \longrightarrow & SO(3) \times SO(3) & \twoheadrightarrow & S^2 \times S^2
 \end{array}$$

We thus have a diffeomorphism

$$SO(4)/(SO(2) \times SO(2)) \cong S^2 \times S^2$$

and the thing on the left is the oriented Grassmannian  $\text{Gr}^+(2,4)$ . Something similar happens for  $\text{Gr}^+(3,4)$ , although it's not as surprising. We have diffeomorphisms

$$\begin{aligned}
 SO(4)/(SO(3) \times SO(1)) &\cong SO(4)/H(D) \\
 &\cong D/G_3 \\
 &= X_3 \\
 &= S^3
 \end{aligned}$$

Note that we also get another non-surprising diffeomorphism  $\text{Gr}^+(2,3) \times \text{Gr}^+(2,3) \cong S^2 \times S^2$  from the above diagram.

As another application, note that if  $P$  is a finite subgroup of  $SO(4)$ , then  $PC$  is also finite, where  $C = \{\pm I\}$  is the center of  $SO(4)$ . The image  $PC/C \cong P/(P \cap C)$  in the quotient  $SO(3) \times SO(3)$  is also finite. Note that  $P \cap C$  is either trivial or  $C$ , so the image we get in  $SO(3) \times SO(3)$  is either isomorphic to  $P$  or is quotient of  $P$  by a central subgroup of order 2. The advantage here is that the finite subgroups of  $SO(3)$  are well-known, and come from

symmetries of regular polygons and the Platonic solids. In particular, a finite subgroup of  $SO(3)$  is isomorphic to a subgroup of one of the following:

$$D_{2n} \qquad A_4 \qquad S_4 \qquad A_5$$

Now suppose the finite group  $P \leq SO(3)$  is perfect and superperfect, meaning  $H_1(P, \mathbb{Z}) = H_2(P, \mathbb{Z}) = 0$ . Then if  $P$  acts fixed-point-freely on  $S^3$ , the quotient would be a 3-manifold that had the same homology of  $S^3$ , but not simply connected (its fundamental group is isomorphic to  $P$ ). Such a space is called a **homology sphere**. As above,  $P/(P \cap C)$  is a perfect subgroup of  $SO(3) \times SO(3)$ , which means it is isomorphic to a perfect subgroup of  $A_5 \times A_5$ . These are pretty limited:

$$A_5 \times \{1\} \qquad \{1\} \times A_5 \qquad \text{diagonal} \qquad A_5 \times A_5$$

Note the first three groups are isomorphic. Although I won't go into the full details here, finite perfect groups have a covering theory similar to the one for path-connected topological spaces. In particular, a finite perfect group  $G$  has a group  $M(G)$  called the **Schur multiplier**, which is isomorphic to  $H_2(G, \mathbb{Z})$ . To every subgroup  $Z \leq M(G)$ , there is a unique (up to isomorphism) perfect group  $G_Z$ , with a central subgroup  $K \cong Z$ , such that

$$G_Z/K \cong G \\ M(G_Z) \cong M(G)/Z$$

In particular, there is a universal covering  $\tilde{G}$ , with  $M(\tilde{G}) = 0$ . These are precisely the finite superperfect groups.

Because  $M(A_5)$  has order 2,  $M(A_5 \times A_5)$  has order 4, and since  $|C| = 2$ , the only way for  $M(P)$  to be trivial is if  $P$  is the universal cover of  $A_5$ , which is  $SL(2, 5)$ . So  $P$  is unique up to isomorphism, assuming it exists in the first place. That is, we don't yet know that there exists a copy of  $SL(2, 5) \leq SO(4)$ , acting fixed-point-freely on  $S^3$ . There is however; in fact we can find  $SL(2, 5) \leq S^3$ , acting then by right multiplication, and such an action is fixed-point free. It's not terribly hard to construct, though computationally tedious. The main idea is to look at a subgroup of  $S^3$  generated by  $j$  and (a suitable conjugate of) a primitive 5th root of unity.

The reason this  $P$  is interesting is that, by the spherical space form theorem, every 3-manifold with finite fundamental group comes from a fixed-point free action of a finite subgroup of  $SO(4)$  on  $S^3$ . Thus, combining the above with this very deep theorem, it turns out that the only homology sphere with finite fundamental group is the one coming from this  $SL(2, 5)$  action.