

Introduction

This short note gives two proofs that the Grassmannian $\text{Gr}^+(2,4)$, of oriented 2-planes in $\mathbb{R}^4$, is diffeomorphic to $S^2 \times S^2$.

Via projective embedding

Every point of $\text{Gr}^+(2,4)$ can be represented by an ordered tuple (u, v) of orthonormal vectors. If (μ, ν) is another ordered basis for the same plane, then the two are related via an element of $GL^+(2, \mathbb{R})$, that is, a matrix A with $\det(A) > 0$. The plan to find the diffeomorphism between $\text{Gr}^+(2,4)$ and $S^2 \times S^2 \cong \mathbb{CP}^1 \times \mathbb{CP}^1$ is to embed $\text{Gr}^+(2,4)$ in a complex projective space first. This is not *too* surprising, since an oriented 2-plane naturally admits a complex structure: u becomes the "real" direction, and v becomes the "imaginary" direction.

We will embed $\text{Gr}^+(2,4)$ into $\mathbb{CP}^3$ via the map ϕ given by

$$\phi : (u, v) \mapsto [u_1 + iv_1 : u_2 + iv_2 : u_3 + iv_3 : u_4 + iv_4] = [u + iv]$$

Here (u, v) is an orthonormal basis for the plane. In order to show this map is well defined, we will modify ϕ so that it first does the Gram-Schmidt process to make any basis an orthonormal basis, and then the map to $\mathbb{CP}^3$ is the same. So for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL^+(2, \mathbb{R})$ we get

$$\begin{aligned} (Au, Av) &\mapsto \left[\frac{au + bv}{\|au + bv\|} + i \frac{cu + dv - \frac{(au+bv) \cdot (cu+dv)}{\|au+bv\|^2} (au + bv)}{\|\cdot\|} \right] \\ &= \left[\frac{au + bv}{\sqrt{a^2 + b^2}} + i \frac{a \det(A)v - b \det(A)u}{\sqrt{a^2 \det(A)^2 + b^2 \det(A)^2}} \right] \\ &= [(au + bv) + i(av - bu)] \\ &= [(a - bi)(u + iv)] \\ &= [u + iv] \end{aligned}$$

Here the third line follows because $\sqrt{\det(A)^2} = \det(A) > 0$ (and this wouldn't work if the determinant was negative). This map is smooth at any point, using the "linear graph" charts that send $(a, b, c, d) \in \mathbb{R}^4$ to the plane spanned by $(u + ax + bw, v + cx + dw)$, with x and w an orthonormal completion of (u, v) to a basis. With these charts and the usual ones for $\mathbb{CP}^3$, our map is represented by a combination of linear maps, the Gram-Schmidt process, and a rational function to go from $\mathbb{CP}^3$ to $\mathbb{C}^3$, and all of those are smooth.

The image $\text{Im}(\phi)$ is not all of $\mathbb{CP}^3$, but instead lands in the hypersurface

$$X = \{[a_1 : a_2 : a_3 : a_4] \mid \sum a_i^2 = 0\}$$

To see this, note that the defining relation for X can be broken down into two equations, for the real and imaginary parts:

$$\Sigma \Re(a_i)^2 = \Sigma \Im(a_i)^2 \quad (1)$$

$$\Re(a) \cdot \Im(a) = 0 \quad (2)$$

Because u and v are orthonormal, both **1** and **2** are satisfied by $\phi(u, v)$, since $\Sigma u_i^2 = \Sigma v_i^2 = 1$ and $u \cdot v = 0$. To see that the image is *all* of X , we can define an inverse:

$$[a_1 : a_2 : a_3 : a_4] \mapsto \left(\frac{\Re(a)}{\|\Re(a)\|}, \frac{\Im(a)}{\|\Im(a)\|} \right)$$

Equation (1) ensures that the norms in the denominator are never zero (so our map is a well-defined smooth map and gives unit vectors), and equation (2) ensure the two vectors we get are orthogonal.

So we have an embedding of $\text{Gr}^+(2, 4)$ into $\mathbb{CP}^3$. The reason to do this is because the hypersurfaces like $X \subset \mathbb{CP}^3$ are well understood, and their topology is determined solely by the quadratic form in their defining equation. To exploit this we'll now define a diffeomorphism $\psi : \mathbb{CP}^3 \rightarrow \mathbb{CP}^3$ that takes X to another hypersurface

$$Y = \{[b_1 : b_2 : b_3 : b_4] \mid b_1 b_4 - b_2 b_3 = 0\}$$

The idea behind ψ is simple: we make the new $b_1 b_4$ be the first half of the quadratic sum defining X , and the new $b_2 b_3$ be the negative of the second half. Explicitly,

$$\psi([a_1 : a_2 : a_3 : a_4]) = [a_1 - ia_2 : a_3 - ia_4 : -a_3 - ia_4 : a_1 + ia_2]$$

Note that for $[a] \in X$, we have $b = \psi([a])$ with $b_1 b_4 = a_1^2 + a_2^2$ and $b_2 b_3 = -(a_3^2 + a_4^2)$ so that the image does lie in Y : $\psi(X) \subset Y$. Conversely, the inverse is also easy to explicitly define:

$$\psi^{-1}([b_1 : b_2 : b_3 : b_4]) = [b_1 + b_4 : i(b_1 - b_4) : b_2 - b_3 : i(b_2 + b_3)]$$

and it's just as easy to see that $\psi^{-1}(Y) \subset X$. So we now have a diffeomorphism between X and Y . What's so special about Y ? Well, it's the image of the *Segre embedding*, a general mechanism for embedding products of projective spaces into higher-dimensional projective spaces. Specifically, we have the map η from $\mathbb{CP}^1 \times \mathbb{CP}^1$ to $\mathbb{CP}^3$ given by

$$\eta([a : b], [c : d]) \mapsto [ac : ad : bc : bd]$$

The image of this map lands in Y since $(ac)(bd) - (ad)(bc) = 0$. As might be expected by now, we also have an explicit inverse from Y to $\mathbb{CP}^1 \times \mathbb{CP}^1$:

$$\eta^{-1}([b_1 : b_2 : b_3 : b_4]) = \begin{cases} [b_1 : b_3] \times [b_1 : b_2] & b_1 \neq 0 \\ [b_2 : b_4] \times [b_1 : b_2] & b_2 \neq 0 \\ [b_1 : b_3] \times [b_3 : b_4] & b_3 \neq 0 \\ [b_2 : b_4] \times [b_3 : b_4] & b_4 \neq 0 \end{cases}$$

The defining equation for Y helps show this is actually an inverse. For example, suppose $a \neq 0$ and $c \neq 0$ for a point in $\mathbb{CP}^1 \times \mathbb{CP}^1$, and $b_1 \neq 0$ for a point in Y . Then

$$\begin{aligned}\eta^{-1}\eta([a:b], [c:d]) &= [ac:bc] \times [ac:ad] \\ &= [a:b] \times [c:d] \\ \eta\eta^{-1}([b_1:b_2:b_3:b_4]) &= [b_1b_1:b_1b_2:b_1b_3:b_1b_4] \\ &= [b_1:b_2:b_3:b_4]\end{aligned}$$

where the last equality follows from $b_2b_3 = b_1b_4$ in Y . Similar reasoning applies for the other cases.

Combining this all together, we have a diffeomorphism $\eta^{-1}\psi\phi$ from our Grassmannian $\text{Gr}^+(2,4)$ to $\mathbb{CP}^1 \times \mathbb{CP}^1 \cong S^2 \times S^2$.

Via Lie groups

The reasoning in the previous section is nice because it is "low-tech" and gives an explicit diffeomorphism. However, it can be hard to motivate the various maps if you don't have an algebraic geometry background. In this section, we instead view our Grassmannian as the quotient of a Lie group, and then find that what we're dealing with is just a couple of Hopf fibrations.

To start us off, we'll note oriented Grassmannians can be given by quotients of a suitable Lie group:

$$\text{Gr}^+(m,n) \cong \frac{SO(n)}{SO(m) \times SO(n-m)}$$

This is *not* a Lie group in general, since the subgroup there is not usually normal. It is the subset of somewhat diagonal matrices in $SO(n)$ that look like $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ for $A \in SO(m)$ and $B \in SO(n-m)$. This equivalence shouldn't be too surprising in the $\text{Gr}^+(2,4)$ case analyzed above: every point looked like the orbit of $SO(2)$ without caring about what happened to the other two vectors in the orthonormal basis (the second $SO(2)$).

We will now decompose $SO(4)$ into a product, and show that the subgroup we're quotienting out by also stays a nice product. $SO(4)$ is the orientation-preserving isometry group of S^3 , which we will regard as the Lie group $SU(2)$. This isomorphism comes from thinking about the sphere as the group $\mathbb{U}$ of unit quaternions. The various representations look like:

$$\begin{aligned}(z, w) &\in S^3 \\ z + wj &\in \mathbb{U} \\ \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} &\in SU(2)\end{aligned}$$

We will write elements of $SU(2)$ as capital letters $(A, B, C, \dots)$ to indicate the matrices we work with, reserving I for the identity matrix. We let $SU(2) \times SU(2)$ act on $SU(2)$ via a diagonal (right) action

$$A \cdot (B, C) \mapsto B^{-1}AC$$

This action is smooth (since it's Lie group multiplication), and it's orientation-preserving since $SU(2)$ is connected: a path from (B, C) to (I, I) shows the map determined by (B, C) is in the same class as the identity map, which is orientation-preserving. To see the action is by isometries, we note that the metric on $SU(2)$ is given by $d(X, Y) = \cos^{-1}(\frac{1}{2} \text{Tr}(X^{-1}Y))$, and so we compute what happens to the trace under action by the element (B, C) :

$$\begin{aligned} \text{Tr}([X \cdot (B, C)]^{-1}Y \cdot (B, C)) &= \text{Tr}(C^{-1}X^{-1}BB^{-1}YC) \\ &= \text{Tr}(C^{-1}X^{-1}YC) \\ &= \text{Tr}(X^{-1}YCC^{-1}) \\ &= \text{Tr}(X^{-1}Y) \end{aligned}$$

and so the action is indeed by isometries. Thus this action gives a map from $SU(2) \times SU(2)$ to $SO(4)$, and this map is a Lie group homomorphism since the action is smooth.

The kernel of this map is easily seen to be just $\{\pm(I, I)\}$. Since this is a finite set, the image of our homomorphism is the same dimension as $SU(2) \times SU(2)$, 6-dimensional. But that's the same dimension as $SO(4)$, so this homomorphism is a submersion. Now submersions are open maps, and $SU(2) \times SU(2)$ is compact and $SO(4)$ is connected, so the homomorphism is *surjective*. Hence we have a Lie group isomorphism

$$SO(4) \cong \frac{SU(2) \times SU(2)}{\pm(I, I)}$$

What's the image of the $SO(2) \times SO(2)$ subgroup under this isomorphism? We can take this one factor at a time. The first $SO(2)$ acts by rotations on the z -part of S^3 , leaving the w -part fixed. In quaternions, we're looking for each θ to solve

$$(a + bj)^{-1}(z + wj)(c + dj) = e^{i\theta}z + wj$$

This becomes much easier to solve once it's realized that b and d must both be 0, since otherwise w will "contribute" to the z -part of the right hand side. And then things become almost trivial when this is combined with the fact that $jx = \bar{x}j$ for all complex numbers x . We have

$$(z + wj) \cdot (e^{-i\frac{\theta}{2}}, e^{i\frac{\theta}{2}}) = e^{i\theta}z + wj$$

In matrices, the image of the first $SO(2)$ factor in $SU(2) \times SU(2)$ is

$$H_1 = \left\{ \begin{pmatrix} \bar{u} & 0 \\ 0 & u \end{pmatrix} \times \begin{pmatrix} u & 0 \\ 0 & \bar{u} \end{pmatrix} \mid u \in S^1 \right\}$$

The second $SO(2)$ factor is similar to determine, and we can see that

$$(e^{-i\frac{\theta}{2}})^{-1}(z + wj)(e^{-i\frac{\theta}{2}}) = z + e^{i\theta}wj$$

And so the image of the second $SO(2)$ factor in $SU(2) \times SU(2)$ is

$$H_2 = \left\{ \begin{pmatrix} \bar{u} & 0 \\ 0 & u \end{pmatrix} \times \begin{pmatrix} \bar{u} & 0 \\ 0 & u \end{pmatrix} \mid u \in S^1 \right\}$$

Let $SU(1) \leq SU(2)$ be the subgroup of diagonal matrices, which represent a circle in S^3 . It's straightforward to see that $H_1 H_2 = SU(1) \times SU(1)$, since for the same $u \in S^1$, a product of the element in H_1 with the corresponding element in H_2 gives the $\bar{u}^2$ element in the first $SU(1) \leq SU(2)$. Similar reasoning shows how to get any element in the second $SU(1)$. This is enough, since clearly $H_1, H_2 \leq SU(1) \times SU(1)$.

Finally, we note that $\pm(I, I) \leq SU(1) \times SU(1)$, and so our isomorphism from above gives us

$$\begin{aligned} \frac{SO(4)}{SO(2) \times SO(2)} &\cong \frac{SU(2) \times SU(2)}{SU(1) \times SU(1)} \\ &\cong \frac{SU(2)}{SU(1)} \times \frac{SU(2)}{SU(1)} \end{aligned}$$

Note that these are *not* group isomorphisms, since the thing on the left is not a group; this is simply an equivalence of coset spaces. On the level of manifolds, $SU(2)/SU(1)$ is the quotient of S^3 by the free action of S^1 ; that is, it's the base space of the Hopf fibration, S^2 . Thus we see that our Grassmanian $\text{Gr}^+(2, 4)$ is actually just the product of two Hopf fibration base spaces, $S^2 \times S^2$.