

Introduction

In this note, we show that the diffeomorphism group $\text{Diff}(M)$ for a compact smooth manifold M is in fact a topological group. To do that, we first build an auxiliary manifold called the **r -jet space**. This is a more general construction, that works for all $C^\infty(M, N)$, and that's how we treat it. It turns out there are multiple topologies that can be put on $C^\infty(M, N)$, and each has its own virtues, but for compact manifolds they all more-or-less coincide. So we pick the most intuitive one.

A note on notation: for a tuple $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{Z}^m$, we write $|\alpha| = m$ and for another m -tuple x , we let

$$x^\alpha = x_1^{\alpha_1} \cdots x_m^{\alpha_m}$$

The utility of these tuples actually shines brightest when applied to partial derivatives; we define

$$\partial^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x^\alpha}$$

Jets

The idea of jets is pretty attractive: a coordinate-free interpretation of Taylor series for smooth manifolds. Of course, even for Euclidean space, there exists distinct smooth functions with the same Taylor series, so trying to build the *entire* series is too restrictive. Instead, we care about truncated Taylor series: chopping off all monomial terms with degree larger than our cutoff.

To start, for two curves $\gamma, \delta : \mathbb{R} \rightarrow N$ into a smooth manifold N , we define r -equivalence of these curves (with $r \geq 0$ an integer), written $\gamma \stackrel{r}{\sim} \delta$, if $\gamma(0) = \delta(0)$ and if for all smooth functions $\eta \in C^\infty(N, \mathbb{R})$,

$$(\eta \circ \gamma)^{(k)}(0) = (\eta \circ \delta)^{(k)}(0)$$

for $k \leq r$. Note that $\gamma \stackrel{0}{\sim} \delta$ just means $\gamma(0) = \delta(0)$. Also note that for $s \leq r$, r -equivalence implies s -equivalence.

Now for $f, g \in C^\infty(M, N)$, we can also define a notion of r -equivalence at a point $x \in M$. We say that $f \stackrel{r}{\sim}_x g$ when for all curves $\gamma : \mathbb{R} \rightarrow M$ with $\gamma(0) = x$, we have $(f \circ \gamma) \stackrel{r}{\sim} (g \circ \gamma)$. To ease notation we will write $j^r f(x) = j^r g(x)$ for this equivalence relation. Just as above, note 0-equivalence just means $f(x) = g(x)$. The set of all equivalence classes will be written $J_x^r(M, N)$, and we can then form the disjoint union

$$J^r(M, N) = \bigsqcup_{x \in M} J_x^r(M, N)$$

We call $J^r(M, N)$ the **r -jets** for M and N . We also highlight here that one can write $J_x^r(M, N)_y$ to emphasize the image of x as well, and that $J_x^r(M, N) = \bigsqcup J_x^r(M, N)_y$. To link this back to

our earlier discussion about truncated Taylor series, we note that being r -equivalent really means that, in any coordinate representation, f and g have the same partial derivatives of order $\leq r$:

PROPOSITION 1: *Let (U, ϕ) be a chart centered at $x \in M$, and (V, ψ) a chart centered at $f(x) \in N$. Then $j^r f(x) = j^r g(x)$ if and only if for all tuples α with $|\alpha| \leq r$ and $1 \leq i \leq n$,*

$$\partial^\alpha \psi^i f \phi^{-1}(0) = \partial^\alpha \psi^i g \phi^{-1}(0)$$

Note that being centered at x and y is only a convenience; clearly this proposition holds for any charts containing x and y by applying translations first.

Proof. First, we show the case for curves: $\gamma \stackrel{r}{\sim} \delta$ if and only if for $1 \leq i \leq n$ and $0 \leq k \leq r$,

$$(\psi^i \circ \gamma)^{(k)}(0) = (\psi^i \circ \delta)^{(k)}(0)$$

One direction is easy, since $\psi^i \in C^\infty(N, \mathbb{R})$. For the other direction, let $\eta \in C^\infty(N, \mathbb{R})$. The chain rule then shows that

$$(\eta \circ \gamma)^{(k)}(0) = ((\eta \circ \psi^{-1}) \circ (\psi \circ \gamma))^{(k)}(0)$$

is a combination of partial derivatives of $\eta \psi^{-1}$ evaluated at 0, and derivatives of $\psi^i \gamma$ evaluated at 0. Since the former are independent of γ and the latter are (by hypothesis) equal for γ and δ , we see that $\gamma \stackrel{r}{\sim} \delta$.

Then for any curve γ with $\gamma(0) = x$ we have that $j^r f(x) = j^r g(x)$ precisely when

$$(\psi^i \circ f \gamma)^{(k)}(0) = (\psi^i \circ g \gamma)^{(k)}(0)$$

But $\psi^i f \gamma = (\psi^i f \phi^{-1})(\phi \gamma)$, and again the chain rule shows this depends on the partial derivatives of the first at 0 and the second at 0. Since the latter do not depend on f , we see that the hypothesis of the proposition implies $j^r f(x) = j^r g(x)$.

Conversely, if $j^r f(x) = j^r g(x)$, we can pick curves $\gamma_a(t) = at$, with $a \in \phi(U)$. Then if we choose $\psi^i \in C^\infty(N, \mathbb{R})$, r -equivalence implies

$$(\psi^i f \phi^{-1} \gamma_a)^{(k)}(0) = (\psi^i g \phi^{-1} \gamma_a)^{(k)}(0)$$

The chain rule once again shows that the k th derivative is given by

$$\sum_{|\alpha|=k} \partial^\alpha \psi^i f \phi^{-1}(0) a^\alpha = \sum_{|\alpha|=k} \partial^\alpha \psi^i g \phi^{-1}(0) a^\alpha$$

Since the a were arbitrary, this implies $\partial^\alpha \psi^i f \phi^{-1}(0) = \partial^\alpha \psi^i g \phi^{-1}(0)$ for each α with $|\alpha| = k$. Since i was arbitrary and this works for $0 \leq k \leq r$, we are done. $\square$

This proposition is the key to putting a topology on $J^r(M, N)$. First, consider $J^r(U, V)$, with $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ open subsets. It's important to call out here that $j^r f(x)$ only depends on the germ of f at x , and using bump functions, it's possible to work with smooth maps defined only on the vicinity of x . Since x has a convex open neighborhood contained in U , every smooth function has a Taylor series, which we can truncate. We see by the proposition that these truncated series represent all equivalence classes, and distinct series give rise to distinct classes. Thus, for $x \in U$ and $y \in V$, the complete set of r -jets in $J_x^r(U, V)_y$ is represented by Taylor polynomials with no constant term of degree at most r :

$$J_x^r(U, V)_y = \left\{ \sum_{1 \leq |\alpha| \leq r} a_\alpha x^\alpha \mid a_\alpha \in \mathbb{R} \right\}$$

where we write $(x_1, \dots, x_m)$ for the coordinates on U , and use α merely as an index for the a . This is naturally bijective with $\mathbb{R}^{\beta(r)}$, with $\beta(r)$ the number of terms in the sum, multiplied by the dimension of the a :

$$\beta(r) = n \left(\binom{m+r}{m} - 1 \right)$$

[Here the -1 comes from not having a constant term in the Taylor series.] Taking disjoint unions we have as sets

$$J^r(U, V) = U \times V \times \mathbb{R}^{\beta(r)}$$

We now declare this to be a diffeomorphism. This allows us to put a topology on $J^r(M, N)$ as follows:

PROPOSITION 2: $J^r(M, N)$ is a smooth manifold of dimension $m + n + \beta(r)$.

Proof. We first define what we want to be the charts for this manifold structure. To that end, let (U, ϕ) and (V, ψ) be charts on M and N as before. For the proof fix $x \in U$ and $y \in V$, and write $u = \phi(x)$ and $v = \psi(y)$. We define

$$\mathcal{W}(U, V) = \bigsqcup_{\substack{x \in U \\ y \in V}} J_x^r(M, N)_y$$

This is one of the charts we want; we define the potential coordinate map by mapping to $J^r(\phi(U), \psi(V))$. To ease notation, we will write $\tilde{f}$ for the trivialization $\psi f \phi^{-1}$. The coordinate map is then

$$\begin{aligned} \theta_{UV} : \mathcal{W}(U, V) &\rightarrow J^r(\phi(U), \psi(V)) \\ j^r f(x) &\mapsto j^r \tilde{f}(u) \end{aligned}$$

We have to show this is a well-defined bijection. Suppose that $j^r f(x) = j^r g(x)$. Choose an arbitrary curve $\gamma : \mathbb{R} \rightarrow \phi(U)$ with $\gamma(0) = u$, and an arbitrary smooth function $\eta \in C^\infty(\psi(V), \mathbb{R})$. Then $\phi^{-1}\gamma$ is a curve with $\phi^{-1}\gamma(0) = x$. $\eta\psi$ is a smooth function on V ; by finding a compact neighborhood of y in V , we can use a bump function to build a smooth

function $\omega \in C^\infty(N, \mathbb{R})$ that still agrees with $\eta\psi$ on this compact neighborhood. Then saying $j^r f(x) = j^r g(x)$ means

$$\begin{aligned}(\omega f \phi^{-1} \gamma)^{(k)}(0) &= (\omega g \phi^{-1} \gamma)^{(k)}(0) \\(\omega \psi^{-1} \tilde{f} \gamma)^{(k)}(0) &= (\omega \psi^{-1} \tilde{g} \gamma)^{(k)}(0) \\(\eta \tilde{f} \gamma)^{(k)}(0) &= (\eta \tilde{g} \gamma)^{(k)}(0)\end{aligned}$$

Here the last equality follows because ω and $\eta\psi$ agree on a neighborhood of y . Since γ and η were arbitrary and this equality holds for $k \leq r$, by definition this means $j^r \tilde{f}(u) = j^r \tilde{g}(u)$.

To see θ_{UV} is injective, if $j^r \tilde{f}(u) = j^r \tilde{g}(u)$, then the previous proposition shows $j^r f(x) = j^r g(x)$. For surjectivity, let $j^r h(u)$ be an arbitrary r -jet in $J^r(\phi(U), \psi(V))$. Find an open ball neighborhood Y of y , and choose a compact neighborhood X of x with $h(X) \subset Y$. Finally, let μ be a diffeomorphism between Y and $\mathbb{R}^n$. Then $\phi^{-1}(X)$ is a compact neighborhood of x , and thus $\mu h \phi : \phi^{-1}(X) \rightarrow \mathbb{R}^n$ can be extended to a map $F : M \rightarrow \mathbb{R}^n$, so that $\psi^{-1} \mu^{-1} F \in C^\infty(M, N)$. But on $\phi^{-1}(X)$ we have

$$\begin{aligned}\widetilde{\psi^{-1} \mu^{-1} F} &= \psi \psi^{-1} \mu^{-1} \mu h \phi \phi^{-1} \\ &= h\end{aligned}$$

Thus $\theta_{UV}(j^r \psi^{-1} \tau^{-1} F(x)) = j^r h(u)$ and so θ_{UV} is a bijection. So our goal now is to show that these charts actually give a manifold structure on $J^r(M, N)$. First, note that if $\{U_i\}$ and $\{V_j\}$ are countable chart covers for M and N , then $\{\mathcal{W}(U_i, V_j)\}$ is a countable chart cover for $J^r(M, N)$, so it's second-countable. For $x \neq \hat{x} \in M$ and $y \neq \hat{y} \in N$, note that there are charts U and $\hat{U}$ separating x and $\hat{x}$, and charts V and $\hat{V}$ separating y and $\hat{y}$. So for $f, g \in C^\infty(M, N)$, $j^r f(x)$ is separated from $j^r f(\hat{x})$ via $\mathcal{W}(U, V)$ and $\mathcal{W}(\hat{U}, V)$. Meanwhile if $g(x) = \hat{y}$, then $j^r f(x)$ is separated from $j^r g(x)$ via $\mathcal{W}(U, V)$ and $\mathcal{W}(U, \hat{V})$. If $g(x) = y$, then $j^r f(x)$ and $j^r g(x)$ both live in $\mathcal{W}(U, V)$ and can be separated there. Thus $J^r(M, N)$ is Hausdorff.

Now $J^r(\phi(U), \psi(V))$ is an open subset of $\mathbb{R}^{m+n+\beta(r)}$. To finish showing $J^r(M, N)$ is a manifold, we need to show that the intersection of two charts remains open, and that the transition maps are smooth. We have

$$\mathcal{W}(U, V) \cap \mathcal{W}(\hat{U}, \hat{V}) = \mathcal{W}(U \cap \hat{U}, V \cap \hat{V})$$

and this maps to an open subset of both $J^r(\phi(U), \psi(V))$ and $J^r(\hat{\phi}(\hat{U}), \hat{\psi}(\hat{V}))$. Finally, let $\sum a_\alpha x^\alpha$ be an element of $J^r(\phi(U), \psi(V))$. Then

$$\theta_{\hat{U}\hat{V}} \theta_{UV}^{-1} \left(\sum a_\alpha x^\alpha \right) = \sum b_\alpha x^\alpha$$

where the b_α are linear combinations of the a_α , $\partial^\alpha \hat{\psi} \psi^{-1}$, and $\partial^\alpha \hat{\phi} \phi^{-1}$, and so this transition is smooth. Thus $J^r(M, N)$ is a smooth manifold. $\square$

In fact, the map π sending $J_x^r(M, N)_y$ to $(x, y) \in M \times N$ actually gives a smooth bundle structure

$$\begin{array}{c} J^r(M, N) \\ \downarrow \pi \\ M \times N \end{array}$$

π is surjective because for any $y \in N$, there is a constant map c_y sending all of M to y . For $(x, y) \in M \times N$, choose charts U and V around x and y . Then $\pi(\mathcal{W}(U, V)) \subset U \times V$ so that π is continuous. Moreover, $\mathcal{W}(U, V)$ gives a local trivialization:

$$\begin{array}{ccc} \mathcal{W}(U, V) & \xrightarrow{\cong} & J^r(\phi(U), \phi(V)) \\ \downarrow \pi & & \downarrow \cong \\ U \times V & & \phi(U) \times \psi(V) \times \mathbb{R}^{\beta(r)} \\ \uparrow & & \downarrow \\ U \times V \times \mathbb{R}^{\beta(r)} & \xleftarrow{\cong} & \phi(U) \times \psi(V) \times \mathbb{R}^{\beta(r)} \end{array}$$

Thus π is smooth, since locally it is just projection. We will write σ for the composition of π and projection to M , and similarly we define τ for the composition of π and the projection to N . [Think of σ as the "source", and τ as the "target".]

There are lots of sections of this bundle. For any $f \in C^\infty(M, N)$, the map $j^r f : M \rightarrow J^r f(M)$ is a bijection, and its inverse σ is continuous and open, thus gives a homeomorphism $x \mapsto j^r f(x)$. For N , note that for any fixed $x \in M$, the map $y \mapsto j^r c_y(x)$ is also a bijection with inverse τ , and thus a homeomorphism.

In fact, for $s \leq r$, there is a bundle map π_{rs} from $J^r(M, N)$ to $J^s(M, N)$, compatible with the bundle over $M \times N = J^0(M, N)$. One can then form the projective limit of these bundles to get an *infinite-dimensional manifold* $J^\infty(M, N)$, but we won't say any more about that.

We can be even more explicit about the open subsets of $J^r(M, N)$. If W is an open subset, then

$$W = \bigcup (W \cap \mathcal{W}(U, V))$$

as U and V range over charts over M and N . Now $W \cap \mathcal{W}(U, V)$ is an open subset of $\mathcal{W}(U, V) \cong \phi(U) \times \psi(V) \times \mathbb{R}^{\beta(r)}$. Since open subsets of charts are charts, and by the definition of the product topology, we see that $W \cap \mathcal{W}(U, V)$ is a union of open subsets that look like

$$\hat{U} \times \hat{V} \times O$$

where $\hat{U} \subset U$, $\hat{V} \subset V$, and O is an open subset of $\mathbb{R}^{\beta(r)}$. Thus W is a union of open subsets that look like this, and we can say that a basis element of $J^r(M, N)$ looks like $\hat{U} \times \hat{V} \times O$ in the chart $\mathcal{W}(\hat{U}, \hat{V})$.

Whitney topology

The Whitney topology we define in this section formalizes the idea that two smooth maps are close if their derivatives are close across the whole manifold. Note that the word "close" is doing some heavy lifting there. We could insist the derivatives are *uniformly* close; that is not the topology we define here. Instead, our definition allows the closeness to vary across charts covering M . Of course, when M is compact (the case we care about), the two ideas coincide. When M is non-compact, however, the topology we define is strictly finer than the "uniform" one, and in fact is so fine as to be non-metrizable.

For $r \geq 0$, and U an open subset of $J^r(M, N)$, define

$$W^r(U) = \{f \in C^\infty(M, N) \mid j^r f(M) \subset U\}$$

Obviously this implies $\sigma(U) = M$, so in local trivializations we're essentially putting restrictions on the derivative values, allowing these restriction to vary from chart to chart. When M is compact, we only need finitely many restrictions, and so we get a uniform restriction anyway.

Because $W^r(U) \cap W^r(V) = W^r(U \cap V)$, these sets $\{W^r(U)\}$ form a basis for a topology on $C^\infty(M, N)$. For $r \leq s$, let $V = \pi_{sr}^{-1}(U)$ be an open subset of $J^s(M, N)$. Then we have $f \in W^r(U) \iff f \in W^s(V)$. Thus the W^s topology is finer than the W^r topology; in particular, the $W^\infty = \cup \{W^r\}$ topology is well-defined.

We'd like to show composition of smooth maps is continuous. First we do it for jets:

PROPOSITION 3: *Composition is well-defined and continuous on r -jets.*

A note about what composition means here: it's defined on the subset where it actually makes sense, namely

$$\mathcal{C} = \{(j, k) \in J^r(M, N) \times J^r(N, P) \mid \tau(j) = \sigma(k)\}$$

Proof. We define composition on $\mathcal{C}$ in the obvious fashion: $(j^r f(x), j^r g(y)) \mapsto j^r(g \circ f)(x)$. To see composition is well-defined, let $j^r f(x) = j^r \hat{f}(x)$ and $j^r g(y) = j^r \hat{g}(y)$. Let $\gamma : \mathbb{R} \rightarrow M$ be a curve with $\gamma(0) = x$ and $\eta \in C^\infty(P, \mathbb{R})$. Then

$$\begin{array}{ll} f \circ \gamma \stackrel{r}{\sim} \hat{f} \circ \gamma & (j^r f(x) = j^r \hat{f}(x)) \\ g \circ f \circ \gamma \stackrel{r}{\sim} g \circ \hat{f} \circ \gamma & (\eta g \in C^\infty(N, \mathbb{R})) \\ g \circ \hat{f} \circ \gamma \stackrel{r}{\sim} \hat{g} \circ \hat{f} \circ \gamma & (j^r g(y) = j^r \hat{g}(y)) \end{array}$$

Thus $j^r g f(x) = j^r \hat{g} \hat{f}(x)$.

To see it's continuous, let $x \in M, y \in N, z \in P$, choose coordinate neighborhoods U, V, W around these, and let $f(x) = y$ and $g(y) = z$. We can show continuity locally by working in

$(\mathcal{W}(U, V) \times \mathcal{W}(V, W)) \cap \mathcal{C}$, and its image under composition, which lies in $\mathcal{W}(U, W)$. All of these have nice images under coordinate maps; we'll drop the coordinate diffeomorphisms since they only add clutter:

$$\begin{aligned} j^r f(x) &\leftrightarrow (x, y, p) \in U \times V \times \mathbb{R}^{\beta(r)} \\ j^r g(y) &\leftrightarrow (y, z, q) \in V \times W \times \mathbb{R}^{\beta(r)} \\ j^r gf(x) &\leftrightarrow (x, z, q \circ p) \in U \times W \times \mathbb{R}^{\beta(r)} \end{aligned}$$

Now if $p = \sum a_\alpha x^\alpha$ and $q = \sum b_\beta y^\beta$, then $q \circ p = \sum b_\beta p^\beta$, which is a gnarly composition of additions and multiplications. In particular, it is continuous. Thus composition is continuous at the arbitrary point $(j^r f(x), j^r g(y))$, and is thus continuous everywhere. $\square$

COROLLARY 4: *For compact manifolds, composition*

$$C^\infty(M, N) \times C^\infty(N, P) \rightarrow C^\infty(M, P)$$

is continuous.

It turns out that compactness of only M is required to make this true, but the proof is considerably more involved. If M is not assumed to be compact, the conclusion is false. See [Mat69] for more details.

Proof. For this proof we write r -jet composition as c . Let W be any open subset of $J^r(M, P)$ and assume $g \circ f \in W^r(W)$, meaning that $j^r(g \circ f)(M) \subset W$. We want to find open subsets $U \subset J^r(M, N)$ and $V \subset J^r(N, P)$ such that

$$\begin{aligned} j^r f(M) &\subset U \\ j^r g(N) &\subset V \\ c((U \times V) \cap \mathcal{C}) &\subset W \end{aligned}$$

Because if that holds, then for F and G with $j^r F(M) \subset U$ and $j^r G(N) \subset V$, we have

$$j^r GF(M) = c((j^r F(M) \times j^r G(N)) \cap \mathcal{C}) \subset c((U \times V) \cap \mathcal{C}) \subset W$$

Thus for $F \in W^r(U)$ and $G \in W^r(V)$ we have $G \circ F \in W^r(W)$. This means composition is continuous because r and W were arbitrary.

Note that $\mathcal{C}$ is closed in $J^r(M, N) \times J^r(N, P)$, since it is the preimage of the diagonal of $N \times N$ under the map $\tau \times \sigma$. Thus $\mathcal{C} \setminus c^{-1}(W)$ is closed as well; let $\mathcal{O}$ stand for its open complement. Because $j^r gf(M) \subset W$, we have $j^r f(M) \times j^r g(N) \subset \mathcal{O}$. Since M and N are compact, so are $j^r f(M)$ and $j^r g(N)$, so by the tube lemma we can find open subsets U and V with

$$j^r f(M) \times j^r g(N) \subset U \times V \subset \mathcal{O}$$

But then $(U \times V) \cap \mathcal{C} \subset c^{-1}(W)$, and we are done. $\square$

Let $\mathcal{I}^r$ be the set of *invertible* elements in $J^r(M, M)$; this means that $\mu \in \mathcal{I}$ when there exists an element $\mu^{-1} \in J^r(M, M)$ such that

$$\begin{aligned}\sigma(\mu) &= \tau(\mu^{-1}) & \tau(\mu) &= \sigma(\mu^{-1}) \\ \mu^{-1} \circ \mu &= j^r \text{id}(\sigma(\mu)) & \mu \circ \mu^{-1} &= j^r \text{id}(\tau(\mu))\end{aligned}$$

The usual argument shows that when μ^{-1} exists, it is unique. Similar to the arguments above, we show inversion is continuous for jets, then that this implies it's continuous for smooth maps.

PROPOSITION 5: *On $\mathcal{I}^r$ the inversion map $\mu \mapsto \mu^{-1}$ is continuous.*

Proof. We prove this locally, where we have maps $U \times V \times \mathbb{R}^{\beta(r)}$ to $V \times U \times \mathbb{R}^{\beta(r)}$. When $r = 0$, the map is just $(u, v) \mapsto (v, u)$, which is continuous. When $r = 1$ is when things become more interesting. A point of $\mathcal{I}^1$ locally looks like

$$(u, v, A) \in U \times V \times \mathbb{R}^{m^2}$$

where A can be thought of as an invertible matrix. The image under inversion is then (v, u, A^{-1}) , which is continuous.

More generally for $r > 1$, an element of $\mathcal{I}^r$ locally looks like $(u, v, p(x))$ where $p(x)$ is a degree r polynomial in m variables and m dimensions. The inverse looks like $(v, u, q(x))$. The Lagrange inversion theorem then expresses the coefficients of q as rational functions of the coefficients of p , and thus inversion is continuous for $r > 1$ as well. $\square$

Note that inversion is a bijective involution, so the previous proposition shows it is in fact a self-homeomorphism on $\mathcal{I}^r$ for all $r \geq 0$.

COROLLARY 6: *Inversion is continuous on $\text{Diff}(M)$.*

Proof. Let $f^{-1} \in W^r(U)$ be a diffeomorphism. Now for all $x \in M$, $j^r f(x) \in \mathcal{I}^r$. Thus $j^r f(M) \subset U \cap \mathcal{I}^r$; we'll call this intersection $\hat{U}$, and note that it's open in $\mathcal{I}^r$ by definition. By the previous proposition and the comment after its proof, the set $\hat{V} = \{\mu^{-1} \mid \mu \in \hat{U}\}$ is also open in $\mathcal{I}^r$, and can then be written as $V \cap \mathcal{I}^r$. Thus $f \in W^r(V)$ and if $g \in W^r(V)$ then $g^{-1} \in W^r(U)$. This shows inversion is continuous, since f , r , and U were arbitrary. $\square$

The previous two corollaries then imply

THEOREM 7: *Under the W^∞ topology, $\text{Diff}(M)$ is a topological group.*

Bibliography

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