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Problems

1 Real Analysis

1.1 Limits and Convergence

[1.1.1] Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function such that there exists $n > 0$ and $x \in \mathbb{R}$ with $f^n(x) = x$. Prove that f has a fixed point.

[1.1.2] Show that for almost every $x \in [0, 2\pi]$, we have

$$\limsup_{n \rightarrow \infty} \sin(nx) = 1$$

[1.1.3] Suppose that $f, g : \mathbb{R}^2 \rightarrow \mathbb{R}$ are both continuously differentiable. For $a, b > 0$ define $F_{a,b} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ via

$$F_{a,b}(x, y) = (af(x, y), bg(x, y))$$

Show that there exists $\epsilon > 0$ such that whenever $a, b < \epsilon$, there is a unique $(x, y) \in \mathbb{R}^2$ with $x^2 + y^2 \leq 1$ and $F_{a,b}(x, y) = (x, y)$.

[1.1.4] Let (X, d) be a complete, non-empty metric space. Let $T : X \rightarrow X$ be a mapping such that the n th iterate T^n is a contraction mapping. Show that T has a unique fixed point.

[1.1.5] Let $C[0, 1]$ be the continuous functions on the unit interval, and let $\{F_n\} \subset C[0, 1]$ be a set of non-negative continuous functions. For $\delta > 0$ define $S_\delta = \{f \in C[0, 1] \mid [0, \delta] \subset f^{-1}(0)\}$, and let $S = \cup S_\delta$ be the set of all functions that are zero on an initial subinterval of $[0, 1]$. Finally define $\Lambda : C[0, 1] \rightarrow \mathbb{R}$ as

$$\Lambda(g) = \lim_{n \rightarrow \infty} \int_0^1 F_n(x)g(x)dx$$

1.2 Differential Equations

Assume that this limit exists for all $g \in C[0, 1]$, and that $\Lambda(f) = 0$ for all $f \in S$. Prove there exists a constant C such that $\Lambda(g) = Cg(0)$ for all $g \in C[0, 1]$.

[1.1.6] Evaluate (with proof)

$$\lim_{n \rightarrow \infty} \int_0^n \left(1 + \frac{x}{n}\right)^n e^{-2x} dx$$

[1.1.7] Let $f : [0, 1] \rightarrow \mathbb{R}$ be a function. Prove that f is Lipschitz, with $|f(x) - f(y)| \leq M|x - y|$ for some $M > 0$ and all $x, y \in [0, 1]$, if and only if there exists a sequence $\{f_n\} \subset C^1([0, 1])$ such that $f_n(x) \rightarrow f(x)$ for all $x \in [0, 1]$, and $\|f'_n\|_\infty \leq M$ for all n .

1.2 Differential Equations

[1.2.1]

- Find a first-order, 2×2 linear homogeneous system of ODEs whose solution space is spanned by $y_1(x) = a \sin(x) + b \cos(x)$ and $y_2(x) = a \cos(x) - b \sin(x)$.
- Find the solution of $y'_1 = y_1 - y_2$ and $y'_2 = 4y_1 - 3y_2$ with initial conditions $y_1(0) = 0$ and $y_2(0) = -1$.

[1.2.2]

- (a) Show that if $\ddot{f} = -f$ on some interval I , then $\dot{f}^2 + f^2$ is constant on I .
- (b) Consider the differential equation $\ddot{g} = -e^g$, with initial conditions $g(0) = 0$ and $\dot{g}(0) = 10$. Find the maximal interval of existence for g , and show that $\dot{g}$ is bounded on this interval.

[1.2.3] Write down a system of linear differential equations satisfied by the two real functions

$$f(\lambda) = \int_0^1 x^{a-1} (1-x)^b (\lambda-x)^c dx$$
$$g(\lambda) = \int_0^1 x^a (1-x)^{b-1} (\lambda-x)^c dx$$

1.2 Differential Equations

Here $\lambda > 1$, $a, b > 0$, and $c \in \mathbb{R}$ is any real number.

[1.2.4] Consider the equation $y'(x) = y(y(x))$ for $x \geq 0$ and $y(0) = 0$.

(a) Show the equation has at least two distinct continuous solutions for $x \geq 0$.

(b) Show only one of these is analytic at $x = 0$.

[1.2.5] Let $p(x)$ be a real polynomial. Assume that all solutions to $f'(t) = p(f(t))$ exist for all $t \in \mathbb{R}$. Show that the degree of $p(x)$ is at most 1.

[1.2.6] Here $C([0, 1])$ will mean the space of continuous functions on the unit interval.

(a) Show that the mapping $I : C([0, 1]) \rightarrow C([0, 1])$ given by

$$(If)(x) = e^x + \frac{1}{2} \int_0^{x^2} f(t) dt$$

is a contraction mapping, with $C([0, 1])$ having the supremum norm.

(b) Show that there exists one and only one smooth function f on $[0, 1]$ satisfying the conditions

$$f'(x) = e^x + xf(x^2)$$

and $f(0) = 1$.

[1.2.7] Consider the homogeneous second order differential equation with smooth coefficients defined on the real line,

$$y'' + p(x)y' + q(x)y = 0$$

Show that all solutions $y = y(x)$ of this equation are monotone functions if and only if $q(x) \equiv 0$.

Hint: Let y_1 and y_2 be two smooth monotone functions on $\mathbb{R}$. Show that if any linear combination of these functions is monotone, then there exists a constant $C \in \mathbb{R}$ such that $y_1'(x) = Cy_2'(x)$.

[1.2.8] Let F be a C^2 function on $\mathbb{R}^3$ satisfying $F_{xx} + F_{yy} + F_{zz} = 0$.

1.2 Differential Equations

(a) Show that F has the mean value property; namely,

$$\frac{4}{3}\pi R^3 F(p) = \int_{B_R(p)} F dV$$

where $B_R(p)$ is the ball of radius R centered around the point $p \in \mathbb{R}^3$.

(b) If F is non-constant, show that F cannot achieve a local maximum at any point $q \in \mathbb{R}^3$.

[1.2.9] A function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is called doubly periodic if $f(x+1, y) = f(x, y+1) = f(x, y)$ for all $(x, y) \in \mathbb{R}^2$. Suppose f is a non-zero, doubly periodic function that is in C^2 . Suppose also that $\Delta f + \lambda f = 0$, where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, and $\lambda \in \mathbb{R}$. Show $\lambda \geq 0$.

[1.2.10] Let $U \subset \mathbb{R}^n$ be a connected bounded open subset such that the boundary $\partial U = \overline{U} \setminus U$ is smooth. Let $f : \overline{U} \rightarrow \mathbb{R}$ be a C^1 function whose restriction to U is a C^2 solution of

$$\Delta f = \lambda f^{2m+1}$$

for a positive integer m and some real constant $\lambda \geq 0$. Recall that $\Delta = \sum \frac{\partial^2}{\partial x_i^2}$. If f is identically zero on ∂U , show that $f \equiv 0$.

[1.2.11] Let $a, b, c \in \mathbb{R}$ with $ac - b^2 > 0$, and let f be a smooth function on $\mathbb{R}^2$ with compact support. Show that the differential equation

$$a \frac{\partial^2 u}{\partial x^2} + 2b \frac{\partial^2 u}{\partial x \partial y} + c \frac{\partial^2 u}{\partial y^2} = f$$

has a smooth solution u .

[1.2.12] Find a solution $u(x, y)$ on $\mathbb{R}^2$ to the initial value problem

$$u_{xx} - u_y = 0$$

with $u(x, 0) = x^6 + x^2 + 1$.

[1.2.13] Let $u(x, y)$ be a solution to the Poisson equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -1$$

1.2 Differential Equations

in the open square $U = \{(x, y) \mid -1 < x, y < 1\}$, with homogeneous boundary conditions (i.e., u is zero on ∂U). Find the minimum and maximum values of u .

[1.2.14] Suppose that $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is a smooth function with compact support.

- (a) Show that there is at most one possible solution $u : \mathbb{R}^n \rightarrow \mathbb{R}$ to the equation $\Delta u = f$ if we insist that u is also differentiable with compact support. Here $\Delta = \sum \frac{\partial^2}{\partial x_i^2}$.
- (b) Show that any solution u of $\Delta u = f$ which is smooth and has compact support must be given by the convolution $f * N$, where N is the Newtonian potential, defined as

$$N(x) = \begin{cases} \frac{\log|x|}{2\pi} & n = 2 \\ \frac{|x|^{2-n}}{n(2-n)\omega_n} & n > 2 \end{cases}$$

Here ω_n is the volume of the unit n -ball in $\mathbb{R}^n$. Also recall that by definition

$$f * N(x) = \int_{\mathbb{R}^n} f(y)N(x - y)dy$$

[1.2.15] Recall that 1-forms on $S^1 = \mathbb{R}/\mathbb{Z}$ are represented by $f(x)dx$, where x is a projection of the global coordinate on $\mathbb{R}$ and $f : \mathbb{R} \rightarrow \mathbb{R}$ is smooth of period 1. Prove that any 1-form θ is cohomologous to a form of the type rdx for some constant $r \in \mathbb{R}$. In other words, there exists $r \in \mathbb{R}$ such that $\theta - rdx$ is an exact 1-form.

[1.2.16] Suppose that $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ is continuous, and satisfies the following Lipschitz condition in the second variable:

$$|F(s, t_1) - F(s, t_2)| \leq M |t_1 - t_2|$$

for some $M \in \mathbb{R}$ and for all $s, t_1, t_2 \in \mathbb{R}$. Consider the differential equation $y' = F(x, y)$ and suppose y_1 and y_2 are two solutions with $y_1(0) = a$ and $y_2(0) = b$.

- (a) Prove that

$$|y_1(x) - y_2(x)| \leq |a - b| + M \left| \int_0^x |y_1(s) - y_2(s)| ds \right|$$

- (b) Show that $\|y_1 - y_2\|_\infty \leq 2|a - b|$ on the interval $\left[-\frac{1}{2M}, \frac{1}{2M}\right]$.

1.3 Fourier Series

[1.3.1] Let $f \in L^2(S^1)$ be a nonzero smooth 2π -periodic function such that

$$\int_0^{2\pi} f(x) dx = 0$$

Let $u(x, t)$ be a solution to the system $u_t = u_{xx}$ with conditions $u(x, 0) = f(x)$ and $u(0, t) = u(2\pi, t)$. Prove that for all $t > 0$, the L^2 norm of $u(\cdot, t)$ is strictly less than the L^2 norm of f .

[1.3.2] Let $S^1 = \mathbb{R}/\mathbb{Z}$, and for $f, g \in L^1(S^1)$, recall the convolution

$$f * g(x) = \int_0^1 f(x - y)g(y) dy$$

Show that there does not exist $f \in L^1(S^1)$ with $f * g = g$ for all $g \in L^1(S^1)$.

[1.3.3] Let $f \in C^1(S^1)$, and suppose $\int_0^1 f(x) dx = 0$. Show that

$$2\pi \|f\|_2 \leq \|f'\|_2$$

[1.3.4] For $0 < r \leq 1$ and $0 < t < 2\pi$, let $f_r(t) = (1 - re^{it})^{-1/4}$, the principal branch of the power function.

(a) Compute the Fourier coefficients of $f_r(t)$ for $0 < r < 1$.

(b) Show that $f_1 \in L^1([0, 2\pi])$.

(c) Compute the Fourier coefficients of $f_1(t)$.

[1.3.5] Let $f \in L^1[0, 2\pi]$, and let $\hat{f}(n)$ denote its n th Fourier coefficient $(2\pi)^{-1} \int_0^{2\pi} f(t) e^{-int} dt$. For $s \geq 0$ define the space

$$H^s = \{f \in L^1[0, 2\pi] \mid \sum_{n \neq 0} |n|^{2s} |\hat{f}(n)|^2 < \infty\}$$

(a) If $s > \frac{1}{2}$, show that $\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty$.

1.3 Fourier Series

- (b) Let F be the periodic extension of f to all of $\mathbb{R}$. Again assuming $s > \frac{1}{2}$, show F agrees almost everywhere with a continuous function.
- (c) Now assume $s > \frac{3}{2}$, and F is as above. Show F agrees almost everywhere with a continuously differentiable function.

[1.3.6] The Fourier coefficients of a function f on the unit circle are given by

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) e^{-int} dt$$

for $n \in \mathbb{Z}$. Are $c_n = (1 + \sqrt{|n|})^{-1}$ the Fourier coefficients of a continuous function?

[1.3.7] Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a C^2 real-valued function, periodic of period 2π , and let its Fourier series be $a_0 + \sum_{n>0} (a_n \cos(nx) + b_n \sin(nx))$. Prove that f is an odd function if and only if $a_n = 0$ for all $n \geq 0$.

[1.3.8] Let H be the Sobolev space on $[0, 1]$: the Hilbert subspace of $L^2([0, 1])$ where

$$\|f\|_H^2 = \sum_{n \in \mathbb{Z}} (1 + n^2) |\hat{f}(n)|^2$$

is finite. Here $\hat{f}(n)$ is the n th Fourier coefficient for f . Show there exists a constant C (that does not depend on $f \in H$) such that $\|f\|_\infty \leq C\|f\|_H$.

[1.3.9] Let $C(S^1)$ be the continuous functions on the circle, and for $k \in \mathbb{Z}$ let $e_k(\theta) = e^{ik\theta}$. Suppose that $I \subset \mathbb{Z}$. Let S be the linear span of $\{e_k \mid k \in I\}$, and let $V \subset C(S^1)$ be given by

$$V = \{f \in C(S^1) \mid \int_{-\pi}^{\pi} f(x) e^{-inx} dx = 0 \text{ for all } n \notin I\}$$

Prove that S is dense in V .

[1.3.10] Let $p(x) = a_0 + a_1x + \dots + a_kx^k$ be a polynomial such that for all $n \in \mathbb{Z}$, we have $p(in) \neq 0$. Let $S^1 = \mathbb{R}/2\pi\mathbb{Z}$, and define an operator $P : C^\infty(S^1) \rightarrow C^\infty(S^1)$ as

$$P(f) = p\left(\frac{\partial}{\partial x}\right) f = a_0 f + a_1 \frac{\partial f}{\partial x} + \dots + a_k \frac{\partial^k f}{\partial x^k}$$

Prove that P is surjective.

[1.3.11] Let $f \in L^2(\mathbb{R})$ and for $t \in \mathbb{R}$ define $f_t(x) = f(x + t)$. Show that

$$\lim_{t \rightarrow 0} \|f - f_t\|_2 = 0$$

where the norm being used is the $L^2(\mathbb{R})$ norm.

1.4 Functional Analysis

[1.4.1] Consider the circle $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ and let $\theta \in S^1$ be an "irrational angle": $\theta/2\pi \notin \mathbb{Q}$. Show that for any continuous function $f \in C(S^1)$ with $\int_0^{2\pi} f dx = 0$, the following sequence for $n > 0$ converges uniformly to zero:

$$S_n(f)(x) = \frac{1}{n} \sum_{k=0}^{n-1} f(x + k\theta)$$

1.5 Measure Theory

[1.5.1] Let $\{f_n\}$ be a sequence of real-valued absolutely continuous functions defined on $[0, 1]$. Assume there are constants M and N such that, for all n and all $x \in [0, 1]$, we have

$$|f_n(x)| \leq M$$

$$\int_0^1 (f'_n(x))^2 dx \leq N$$

Prove that $\{f_n\}$ has a uniformly convergent subsequence.

[1.5.2] Let (X, μ) be a σ -finite measure space with $\mu(X) > 0$. Prove there exists a probability measure ν on X (i.e., $\nu(X) = 1$), such that μ is absolutely continuous with respect to ν , and vice versa.

1.5 Measure Theory

- [1.5.3] Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a measurable compactly supported function, and assume for all $n \geq 0$ we have

$$\int_{\mathbb{R}} f(x) x^n dx = 0$$

Prove that f is zero almost everywhere.

- [1.5.4] Let μ be Lebesgue measure on $\mathbb{R}^n$ and let $\{E_n\}$ be a collection of measurable subsets in $\mathbb{R}^n$. Assume that

$$\sum_{n>0} \mu(E_n) < \infty$$

Let S be the points in $\mathbb{R}^n$ that are contained in infinitely many of the E_i . Prove that $\mu(S) = 0$.

- [1.5.5] Give, with proof, an explicit example of an open set $U \subset [0, 1]$, such that the characteristic function χ_U is not Riemann integrable.

- [1.5.6] Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be any function, and suppose it is *locally measurable*, meaning that for every $x \in \mathbb{R}$, there is an open neighborhood U_x containing x such that f restricted to U_x is measurable. Prove or disprove that f is measurable.

- [1.5.7] Let (X, μ) be a measure space and suppose $\mu(X) < \infty$. Let S be the equivalence classes of measurable functions on X , where f is equivalent to g when $f = g$ almost everywhere. Define

$$\rho(f) = \int_X \frac{|f|}{1 + |f|} d\mu$$

Prove that $d(f, g) = \rho(f - g)$ is a metric on S , and that $f_n \rightarrow f$ in this metric if and only if $f_n \rightarrow f$ in measure.

- [1.5.8] Let T be a volume preserving homeomorphism of a bounded domain D in Euclidean space. Show that for any neighborhood U of any point $p \in D$, there exists a point $x \in U$ which returns to U under iterations of T ; that is, there exists $n > 0$ such that $T^n(x)$ belongs to U .

- [1.5.9] Let $S \subset [0, 1]$ be the subset of the unit interval defined by the following property: $x \in S$ precisely when the first 2 in the decimal expansion of S precedes the first 3 in the decimal expansion of x . Prove that S is Lebesgue measurable and find its measure.

1.6 TBM

- [1.6.1] Let $\{a_1, \dots, a_k\}$ be a finite sequence of points in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$. Define the complex function

$$B(z) = \prod_{i=1}^k \frac{z - a_i}{1 - \bar{a}_i z}$$

- (a) Prove that B maps the unit circle $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ to itself.
 (b) If $B(0) = 0$, prove that for all continuous functions $g \in C(S^1)$, we have

$$\int_0^{2\pi} g(e^{it}) dt = \int_0^{2\pi} g(B(e^{it})) dt$$

Solutions

1 Real Analysis

1.1 Limits and Convergence

[1.1.1] This is obvious if $n = 1$, so we assume $n > 1$ and that $f^k(x) \neq x$ for $k < n$. Suppose $f(x) > x$. Then since $f^n(x) = x < f(x)$, there must exist $1 \leq k < n$ such that $f^{k+1}(x) < f^k(x)$; pick k to be the minimal such index. Then the function $g(x) = f(x) - x$ is positive at x , but negative at $f^k(x)$. The Intermediate Value Theorem then shows there is a $y \in [x, f^k(x)]$ such that $g(y) = 0$, and thus y is a fixed point of f . Similar reasoning applies when $f(x) < x$.

[1.1.2] Let $x \in [0, 2\pi]$ not be a rational multiple of π . Note that the set of rational multiples of π has measure zero, so if we show $\limsup \sin(nx) = 1$, then we will be done. Because x is not a rational multiple of π , the ratio $x/(2\pi)$ is irrational. By Kronecker's theorem, the multiples $\{nx\}$, treated modulo 2π , are dense in the interval $[0, 2\pi]$. Thus there is an increasing sequence of integers $n_1, n_2, \dots$ such that $n_i x \rightarrow \frac{\pi}{2} \pmod{2\pi}$. By continuity of $\sin$, we see then that $\sin(n_i x) \rightarrow 1$. Because $\sin$ is always bounded above by 1, this shows that $\limsup \sin(nx) = 1$.

[1.1.3] On the compact subset $\mathbb{D} = \{(x, y) \mid x^2 + y^2 \leq 1\}$, f and g are bounded, so we can find M_f and M_g such that

$$\begin{aligned} |f(x, y)| &\leq M_f & (x, y) &\in \mathbb{D} \\ |g(x, y)| &\leq M_g & (x, y) &\in \mathbb{D} \end{aligned}$$

Then we see that

$$\|F_{a,b}(x, y)\|^2 \leq a^2 |f(x, y)|^2 + b^2 |g(x, y)|^2 \leq a^2 M_f^2 + b^2 M_g^2$$

For $(x, y) \in \mathbb{D}$. We can thus choose $\epsilon > 0$ small enough such that for $a, b < \epsilon$, $F_{a,b}(\mathbb{D}) \subset \mathbb{D}$. Now f and g are also Lipschitz on $\mathbb{D}$, say with constants L_f and L_g . Let $u, v \in \mathbb{D}$. In the L^1 norm then, we have

$$\|F_{a,b}(u) - F_{a,b}(v)\|_1 = a|f(u) - f(v)| + b|g(u) - g(v)| \leq (aL_f + bL_g)\|u - v\|_1$$

1.1 Limits and Convergence

So again, by making ϵ small enough (with $a, b < \epsilon$), we can make $F_{a,b}$ a contraction mapping on $\mathbb{D}$. It thus has a unique fixed point by the Banach Fixed Point Theorem.

[1.1.4] Let z be the unique fixed point of T^n , which exists by the Banach Fixed Point Theorem. Note that any fixed point of T is a fixed point of T^n , and thus T has at most one fixed point. But

$$T^n(T(z)) = T(T^n(z)) = T(z)$$

so that $T(z)$ is also a fixed point of T^n , thus by uniqueness $T(z) = z$.

[1.1.5] We define the constant function $\mathbf{1} \in C[0, 1]$ and let $C = \Lambda(\mathbf{1})$, which exists by hypothesis. First assume that $h \in C[0, 1]$ is such that $h(0) = 0$, and we prove that $\Lambda(h) = 0$. To see that, let $\epsilon > 0$ be arbitrary, and choose $\delta > 0$ such that $|h(x)| < \epsilon$ whenever $x \in [0, 2\delta]$. Now let $\gamma \in C[0, 1]$ be a "cutoff" function, such that:

- $\gamma \equiv 0$ on $[0, \delta]$.
- $\gamma \equiv 1$ in $[2\delta, 1]$.
- γ linearly interpolates between 0 and 1 on $[\delta, 2\delta]$.

Then $h\gamma \in S_\delta \subset S$, so $\Lambda(h\gamma) = 0$. But $\|h - h\gamma\|_\infty < \epsilon$, and thus

$$-\epsilon\mathbf{1} \leq h - h\gamma \leq \epsilon\mathbf{1}$$

Because the F_n are non-negative, we can multiply the above by F_n , then integrate, then take limits, to see that $-\epsilon C \leq \Lambda(h - h\gamma) \leq \epsilon C$. Since $\epsilon > 0$ was arbitrary, this shows $\Lambda(h) = \Lambda(h - h\gamma)$ is zero.

Now for an arbitrary $g \in C[0, 1]$, let $\hat{g} = g - g(0)\mathbf{1}$. From the above $\Lambda(\hat{g}) = 0$, so we have

$$\Lambda(g) = \Lambda(\hat{g}) + \Lambda(g(0)\mathbf{1}) = g(0)\Lambda(\mathbf{1}) = Cg(0)$$

[1.1.6] First, note that for $u \geq 0$ we have $\ln(1 + u) \leq u$. Setting $u = x/n$ and exponentiating then gives

$$\left(1 + \frac{x}{n}\right)^n \leq e^x$$

Of course this sequence converges to e^x for all x . Now define

$$f_n(x) = \left(1 + \frac{x}{n}\right)^n e^{-2x} \chi_{[0, n]}$$

1.2 Differential Equations

Then we're trying to find $\lim_{n \rightarrow \infty} \int_0^\infty f_n dx$. Since every $x \in [0, \infty)$ is eventually in $[0, n]$ for big enough n , we see that $f_n(x) \rightarrow e^{-x}$ for all $x \geq 0$. From the above, we also have $f_n(x) \leq e^{-x}$ for all $n > 0$ and $x \geq 0$. Since $\int_0^\infty e^{-x} dx = 1$, the Dominated Convergence Theorem shows

$$\lim_{n \rightarrow \infty} \int_0^\infty \left(1 + \frac{x}{n}\right)^n e^{-2x} dx = \lim_{n \rightarrow \infty} \int_0^\infty f_n(x) dx = \int_0^\infty \lim_{n \rightarrow \infty} f_n(x) dx = \int_0^\infty e^{-x} dx = 1$$

[1.1.7] First suppose that the sequence $\{f_n\}$ of continuously differentiable functions exists with the given properties. Then for $x, y \in [0, 1]$ with $x < y$ we have

$$f_n(y) = f_n(x) + \int_x^y f'_n(t) dt \leq f_n(x) + M(y - x)$$

Thus $|f_n(y) - f_n(x)| \leq M|y - x|$. The left-hand side is continuous, so taking the limit we see that $|f(y) - f(x)| \leq M|y - x|$, as desired.

Conversely, suppose f is Lipschitz with constant M . Then f is absolutely continuous, and thus has a derivative f' almost everywhere in $[0, 1]$, with

$$f(x) = f(0) + \int_0^x f'(t) dt$$

Since $f' \in L^1([0, 1])$ and $C([0, 1])$ is dense there, we can find a sequence $\{g_n\}$ that converges to f' in $L^1([0, 1])$. Now the Lipschitz condition shows $\|f'\|_\infty \leq M$, so we can assume $\|g_n\|_\infty \leq M$ as well (simply by capping g_n to have range in $[-M, M]$, which doesn't change continuity and only strengthens convergence). Define

$$f_n(x) = f(0) + \int_0^x g_n(t) dt$$

Because g_n is continuous, we see that $f_n \in C^1([0, 1])$, with $f'_n(x) = g_n(x)$. Thus $\|f'_n\|_\infty \leq M$, and we have for $x \in [0, 1]$

$$|f_n(x) - f(x)| = \left| \int_0^x (g_n(t) - f'(t)) dt \right| \leq \int_0^x |g_n(t) - f'(t)| dt \leq \|g_n - f'\|_1$$

This difference tends to zero, so $f_n(x) \rightarrow f(x)$. Thus $\{f_n\}$ is our desired sequence.

1.2 Differential Equations

[1.2.1]

1.2 Differential Equations

- Note that $y_1'(x) = y_2(x)$ and similarly $y_2'(x) = -y_1(x)$; these together give our system of equations.
- We can write these equations as $y' = Ay$, where

$$A = \begin{pmatrix} 1 & -1 \\ 4 & -3 \end{pmatrix}$$

We also have $y(0) = (0, -1)^T$. A solution to this equation is given by $y(x) = e^{Ax}y(0)$. To compute that exponential, we first find the eigenvalues of A , which are $\{-1, -1\}$. Then if we write $A = I + N$, we see that N is nilpotent with $N^2 = 0$. Thus we have

$$e^{Ax} = e^{(-I+N)x} = e^{-Ix}e^{Nx} = e^{-x}(I + Nx) = e^{-x} \begin{pmatrix} 1+2x & -x \\ 4x & 1-2x \end{pmatrix}$$

Multiplying this with our initial condition vector gives the final solution

$$\begin{aligned} y_1(x) &= xe^{-x} \\ y_2(x) &= (2x - 1)e^{-x} \end{aligned}$$

[1.2.2]

- (a) We take the derivative of $\dot{f}^2 + f^2$, showing it's zero, so that the expression is constant on the connected interval I :

$$\begin{aligned} (\dot{f}^2 + f^2)' &= 2\dot{f}\ddot{f} + 2f\dot{f} \\ &= 2\dot{f}(\ddot{f} + f) \\ &= 0 \end{aligned}$$

- (b) We multiply both sides by $\dot{g}$ to get the integrable equation $\ddot{g}\dot{g} = -e^g\dot{g}$, which becomes

$$\dot{g}^2 = C - 2e^g$$

From the initial conditions, $C = 102$. Note this already shows $|\dot{g}| \leq \sqrt{102}$, so $\dot{g}$ is bounded wherever g is defined. Now by Picard-Lindelof, there is a solution g defined in some open interval around 0. Since $\dot{g}$ is bounded, then function grows at a bounded rate, and thus never escapes to $\pm\infty$ in finite time:

$$|g(t)| = \left| \int_0^t \dot{g} \, dt \right| \leq \int_0^t |\dot{g}| \, dt \leq |t|\sqrt{102}$$

This shows the solution g is actually defined on all of $\mathbb{R}$.

1.2 Differential Equations

[1.2.3] Let $h_\lambda(x) = x^a(1-x)^b(\lambda-x)^c$. Note that $a, b > 0$ implies that h_λ is defined and continuous on the entire interval $[0, 1]$. By the fundamental theorem of calculus then, we see

$$\int_0^1 h'_\lambda(x) dx = h_\lambda(1) - h_\lambda(0) = 0 - 0 = 0$$

Performing the differentiation yields

$$0 = af(\lambda) - bg(\lambda) - c \int_0^1 x^a(1-x)^b(\lambda-x)^{c-1} dx$$

We can rewrite that last integral for both f and g . First, we have

$$\begin{aligned} c \int_0^1 x^a(1-x)^b(\lambda-x)^{c-1} dx &= c \int_0^1 (\lambda - (\lambda-x))x^{a-1}(1-x)^b(\lambda-x)^{c-1} dx \\ &= c\lambda \int_0^1 x^{a-1}(1-x)^b(\lambda-x)^{c-1} dx - c \int_0^1 x^a(1-x)^b(\lambda-x)^c dx \\ &= \lambda f'(\lambda) - cf(\lambda) \end{aligned}$$

We thus get the linear equation $af - bg = \lambda f' - cf$. Similar calculation works for g [using $1-x = (1-\lambda) + (\lambda-x)$]. In the end, after rearranging, we get

$$\begin{aligned} f'(\lambda) &= \frac{a+c}{\lambda}f(\lambda) - \frac{b}{\lambda}g(\lambda) \\ g'(\lambda) &= \frac{a}{1-\lambda}f(\lambda) - \frac{b+c}{1-\lambda}g(\lambda) \end{aligned}$$

[1.2.4]

(a) $y = 0$ is one solution. Another can be gotten by branching based on sign:

$$y(x) = \begin{cases} -x^2 & x \geq 0 \\ -2\sqrt{-x} & x \leq 0 \end{cases}$$

Here we see $y(y(x)) = -2x = y'(x)$ for $x \geq 0$. Since the two branches agree at $x = 0$, the function is continuous.

(b) Suppose a continuous solution $y(x)$ is analytic at $x = 0$, so that we can expand $y(x)$ into a convergent Taylor series in some neighborhood $U \subset \mathbb{R}$ of the origin. Since $y(0) = 0$, the series in U takes the form

$$y(x) = \sum_{n=1}^{\infty} c_n x^n$$

1.2 Differential Equations

We can differentiate the above term-by-term to get a series for $y'(x)$, or we can compose it with itself to get a series for $y(y(x))$ (possibly shrinking U so that $y(U) \subset U$). This gives

$$\begin{aligned} y'(x) &= c_1 + 2c_2x + \cdots + nc_nx^{n-1} + \cdots \\ y(y(x)) &= c_1^2x + (c_1c_2 + c_2c_1^2)x^2 + \cdots + p_{n-1}(c_1, \dots, c_{n-1})x^{n-1} + \cdots \end{aligned}$$

Here the $\{p_m\}$ are polynomials in the specified variables, with $p_m(0, \dots, 0) = 0$. Looking at constant terms, we see that equality of these two series gives $c_1 = 0$. Inductively, if we've shown $c_i = 0$ for $0 \leq i < n$, then $p_{n-1}(0, \dots, 0) = 0$, so that $nc_n = 0$; in other words, $c_n = 0$. So by induction we can show that all $c_i = 0$, and $y(x)$ must be identically zero in U .

Now let $X = y^{-1}(0)$, which is a closed subset of $[0, \infty)$. If $z \in X$, then by continuity there is a $\delta > 0$ such that y takes $(z - \delta, z + \delta)$ into U from above. Thus for x with $|x - z| < \delta$ we have $y'(x) = y(y(x)) \in y(U) = \{0\}$. Thus y' is zero on the interval $(z - \delta, z + \delta)$, and y is thus constant on this interval. But $y(z) = 0$, so we see that the open neighborhood $(z - \delta, z + \delta)$ is contained in X . Since $z \in X$ was arbitrary, we see X is an open subset too. By connectedness, and the fact $0 \in X$, we have $X = [0, \infty)$, so that y is identically zero.

[1.2.5] Clearly if $p(y)$ is degree 0, there is a linear solution that exists for all $t \in \mathbb{R}$, and if $p(y) = ay + b$ then $y = Ce^{at} - \frac{b}{a}$ is the family of solutions, and they all exist for all $t \in \mathbb{R}$. If the degree of $p(y)$ is greater than 1, choose a y_0 where $p(y) \neq 0$ for $y \geq y_0$. Consider a solution with initial value condition $y(0) = y_0$. From the solution

$$t = \int_{y_0}^y \frac{dy}{p(y)}$$

we have two cases, noting that $L = \lim_{y \rightarrow \infty} \int_{y_0}^y \frac{dy}{p(y)}$ is finite because the degree of $p(y)$ is at least 2:

- If $p(y) > 0$ for $y \geq y_0$ then $t < L$, and so the solution cannot exist for all t , since it doesn't exist at $t = L$.
- If $p(y) < 0$ for $y \geq y_0$, then $t > L$, and so the solution cannot exist for all t , since it doesn't exist at $t = L$.

[1.2.6]

(a) For $f, g \in C^0([0, 1])$, we have

$$\begin{aligned} \|If - Ig\|_\infty &= \max_{x \in [0, 1]} \left| \left(e^x + \frac{1}{2} \int_0^{x^2} f(t) dt \right) - \left(e^x + \frac{1}{2} \int_0^{x^2} g(t) dt \right) \right| \\ &= \frac{1}{2} \max_{x \in [0, 1]} \left| \int_0^{x^2} (f(t) - g(t)) dt \right| \\ &\leq \frac{1}{2} \max_{x \in [0, 1]} \int_0^{x^2} |f(t) - g(t)| dt \\ &\leq \frac{1}{2} \int_0^1 \|f - g\|_\infty dt \\ &= \frac{1}{2} \|f - g\|_\infty \end{aligned}$$

(b) First note that

$$(If)'(x) = e^x + xf(x^2)$$

So if f is the (unique) fixed point of I , then $f = If$ is C^1 , and satisfies the differential equation. Meanwhile, if g is a solution to the differential equation, then $(Ig)'(x) = g'(x)$, so that $Ig(x) = g(x) + C$ for some constant C . However, note that $Ig(0) = 1$ since the integral in the definition is 0, and since $g(0) = 1$, we see that $C = 0$ and g is a fixed point of I , so that $g = f$. All that's left is to show that f is smooth. But since $f \in C^0([0, 1])$, then

$$f'(x) = (If)'(x) = e^x + xf(x^2)$$

is continuous, so $f \in C^1([0, 1])$. The inductive step is the same: if $f \in C^r([0, 1])$, then $f'(x) = (If)'(x) \in C^r([0, 1])$, so that $f \in C^{r+1}([0, 1])$.

[1.2.7] We will use throughout a different characterization of monotonicity for a C^1 function: the derivative never changes sign. First let's establish the hint. Suppose that no such constant exists, and we will establish a contradiction. This means that there exists two distinct points $x_1, x_2 \in \mathbb{R}$ such that $y'_1(x_1)y'_2(x_2) \neq y'_1(x_2)y'_2(x_1)$. Thus the determinant of

$$\begin{pmatrix} y'_1(x_1) & y'_2(x_1) \\ y'_1(x_2) & y'_2(x_2) \end{pmatrix}$$

is nonzero. It is thus an invertible matrix, so we can find constants $c_1, c_2 \in \mathbb{R}$ such that

$$\begin{aligned} c_1 y'_1(x_1) + c_2 y'_2(x_1) &= 1 \\ c_1 y'_1(x_2) + c_2 y'_2(x_2) &= -1 \end{aligned}$$

But then $y(x) = c_1 y_1(x) + c_2 y_2(x)$ is supposed to be monotone, but its derivative changes sign. This contradiction shows the constant $C \in \mathbb{R}$ does indeed exist.

1.2 Differential Equations

Now assume all solutions to our differential equation are monotone. The solutions span a 2-dimensional space, say with basis $\{y_1(x), y_2(x)\}$. By the hint we just proved, we can find $C \in \mathbb{R}$ such that $y(x) = y_1(x) - Cy_2(x)$ is constant. By linear independence, this constant (call it K) is nonzero. Plugging into our differential equation gives $Kq(x) = 0$, so that $q(x) \equiv 0$, as desired.

Conversely, if $q(x) \equiv 0$, then we can use an integrating factor to solve our ODE for y' :

$$y'(x) = y'(0)e^{-\int_0^x p(t)dt}$$

Since the exponential is always positive, we see that $y'(x)$ has the same sign as $y'(0)$ for all $x \in \mathbb{R}$, meaning $y(x)$ is monotone.

[1.2.8]

- (a) We will assume without loss of generality that p is the origin, and write B_R for $B_R(0)$. From Stokes's theorem, we have

$$\int_{\partial B_R} F_x dy \wedge dz + F_y dz \wedge dx + F_z dx \wedge dy = \int_{B_R} (F_{xx} + F_{yy} + F_{zz}) dV = 0$$

Switching to spherical coordinates, with

$$\begin{aligned} x &= R \sin(\theta) \cos(\phi) \\ y &= R \sin(\theta) \sin(\phi) \\ z &= R \cos(\theta) \end{aligned}$$

Then the 2-forms are given by

$$\begin{aligned} dx \wedge dy &= R^2 \sin(\theta) \cos(\theta) d\theta \wedge d\phi \\ dz \wedge dx &= R^2 \sin^2(\theta) \sin(\phi) d\theta \wedge d\phi \\ dy \wedge dz &= R^2 \sin^2(\theta) \cos(\phi) d\theta \wedge d\phi \end{aligned}$$

So our integral over the ball boundary can then be written as

$$\begin{aligned} 0 &= \int_0^{2\pi} \int_0^\pi R^2 \sin(\theta) (F_x \sin(\theta) \cos(\phi) + F_y \sin(\theta) \sin(\phi) + F_z \cos(\theta)) d\theta d\phi \\ &= R^2 \int_0^{2\pi} \int_0^\pi \sin(\theta) \frac{\partial F}{\partial R} d\theta d\phi \end{aligned}$$

Now write $G(R) = T(R) \int_{B_R} F dV$, where $T(R) = \frac{3}{4\pi R^3}$ is the inverse of the volume of B_R . In spherical coordinates, we have

$$\begin{aligned} G(R) &= T(R) \int_0^R \int_0^{2\pi} \int_0^\pi r^2 \sin(\theta) F d\theta d\phi dr \\ G'(R) &= \frac{-3G(R)}{R} + T(R) \int_0^{2\pi} \int_0^\pi R^2 \sin(\theta) F d\theta d\phi \end{aligned}$$

Calling the second term $H(R)$ we then have

$$\begin{aligned} H'(R) &= \frac{-H(R)}{R} + T(R)R^2 \int_0^{2\pi} \int_0^\pi \sin(\theta) \frac{\partial F}{\partial R} d\theta d\phi \\ &= \frac{-H(R)}{R} \end{aligned}$$

which follows because the integral is zero by the above. We can now solve some simple differential equations to show

$$G(R) = A + BR^{-3}$$

If we can show $\lim_{R \rightarrow 0} G(R) = F(0)$, then we are done, because the limit existing shows $B = 0$ and the limit itself shows $G(R) = F(0)$. But the limit is easy to see from the following inequality:

$$|G(R) - F(0)| \leq \sup_{x \in B_R} |F(x) - F(0)|$$

Because F is continuous at 0, the supremum above converges to 0 as $R \rightarrow 0$. Therefore $\lim_{R \rightarrow 0} G(R) = F(0)$.

- (b) Suppose F achieved a local maximum at a point q , with $F(q) = M$. Let $S = \{p \in \mathbb{R}^3 \mid F(p) = M\}$. Then $S \subset \mathbb{R}^3$ is closed and non-empty; we will show it is also open, so that $F \equiv M$ is constant, a contradiction. To that end, let $p \in S$, and note that $F(x) \leq M$ in some ball centered at p , say $B_R(p)$. If we let V be the volume of $B_R(p)$, then by part (a) we have

$$M = F(p) = \frac{1}{V} \int_{B_R(p)} F dV$$

But then the integral of $(M - F(x))$ over $B_R(p)$ is 0. This is a non-negative expression, so this implies $F(x) = M$ for all $x \in B_R(p)$. This shows $B_R(p) \subset S$, and since p was arbitrary, S must be open, as desired.

[1.2.9] Let $C \in \mathbb{R}^2$ be the unit square. Then $f(C) = f(\mathbb{R}^2)$, and by compactness, f attains a global maximum M and global minimum m . Since f is non-zero, we can divide into two cases:

- Suppose $M > 0$, and choose a point $p \in \mathbb{R}^2$ where $f(p) = M$. By the second derivative test, the Hessian at p must have non-positive eigenvalues, so that its trace is non-positive. But its trace is $\Delta f(p)$. Thus we have

$$\lambda M = \lambda f(p) = -\Delta f(p) \geq 0$$

Since $M > 0$, we see $\lambda \geq 0$ as well.

- Now suppose $m < 0$, and follow the same steps as above. In this case, the Hessian must have non-negative eigenvalues, so that its trace is non-negative. Then

$$\lambda m = \lambda f(p) = -\Delta f(p) \leq 0$$

Since $m < 0$, this again implies $\lambda \geq 0$.

[1.2.10] Since $\bar{U}$ is compact, f attains a maximum and minimum on $\bar{U}$. If we can show it does not have a positive maximum – nor a negative minimum – on the interior U , then it must be that its maximum and minimum are attained on ∂U . Since it is identically zero there, we must have $f \equiv 0$ on all of U .

First suppose that f attains a maximum $M > 0$ on an interior point $p \in U$. At this point, the Hessian must have non-positive eigenvalues, so that its trace is non-positive. But its trace is $\Delta f(p)$. Evaluating our PDE at p gives

$$0 \geq \Delta f(p) = \lambda f^{2m+1}(p) = \lambda M^{2m+1}$$

This is only possible if $\lambda = 0$, and thus $\Delta f = 0$ is harmonic. But a harmonic function cannot achieve a maximum on an interior point unless it is constant, and that contradicts our assumption that $M > 0$ since f is zero on ∂U .

Similarly, suppose f attains a minimum $M < 0$ on an interior point $p \in U$. Here the Hessian has non-negative eigenvalues, so that its trace is non-negative. So once again we get

$$0 \leq \Delta f(p) = \lambda f^{2m+1}(p) = \lambda M^{2m+1}$$

Since $M < 0$ and $2m + 1$ is odd, this is only possible if $\lambda = 0$, and again we have that f is harmonic. Just as before, it must be constant to achieve its minimum at p , and that contradicts $M < 0$.

We've thus shown that its global maximum and minimum must both be 0, and thus f is constant (and zero).

[1.2.11] We define the coefficient matrix

$$H = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

From the hypotheses, $\det(H) > 0$. If the eigenvalues of H are negative, we can negate our entire original equation to assume H is positive definite. Since H is symmetric we can write it as $H = MM^T$ with M an invertible matrix (this is a consequence of the Spectral Theorem). M defines a change of variables $z = M^{-1}(x, y)$, transforming our original equation into the Poisson equation

$$\Delta_z v(z) = g(z)$$

1.2 Differential Equations

with $v(z) = u(Mz)$ and $g(z) = f(Mz)$. We can solve this in the usual way (say by convolving with the Newtonian potential; see [1.2.14]). Since f (hence g) has compact support, our solution $v(z)$ is smooth. Then $u(x, y) = v(z)$ is a smooth solution to our original equation.

[1.2.12] Let $f(x) = x^6 + x^2 + 1$. We will assume that $u(x, y)$ is analytic in a neighborhood of $y = 0$ and look at the Taylor series:

$$u(x, y) = \sum_{i=0}^{\infty} \frac{y^i}{i!} \frac{\partial^i u}{\partial y^i}(x, 0) = \sum_{i=0}^{\infty} \frac{y^i}{i!} \frac{\partial^{2i} u}{\partial x^{2i}}(x, 0) = \sum_{i=0}^{\infty} \frac{y^i}{i!} f^{(2i)}(x)$$

Here we're using that u is analytic in this neighborhood, so we can reorder mixed partial derivatives (and we're repeatedly applying that $u_y = u_{xx}$). Since $f(x)$ is a polynomial, this sum is finite and we can compute that

$$u(x, y) = x^6 + 30x^4y + 180x^2y^2 + 120y^3 + x^2 + 2y + 1$$

It's easy to check directly that $u_y = u_{xx}$.

[1.2.13] Let m be the minimum and M the maximum for u in U . From $\Delta u = -1 < 0$, we know u is superharmonic, and thus by the Minimum Principle it cannot achieve its minimum at an interior point. Thus its minimum is achieved on the boundary, so that $m = 0$.

Now let $v(x, y)$ be a solution to $\Delta v = -1$ without concern for boundary conditions; for example, $v(x, y) = -\frac{1}{4}(x^2 + y^2)$. Then by superposition we can write $u(x, y) = v(x, y) + h(x, y)$, where

$$\Delta h = \Delta u - \Delta v = 0$$

Thus h is a harmonic function on U , and on the boundary (since $u \equiv 0$ on ∂U) we have $h(x, y) = -v(x, y)$. Since h is harmonic, it attains its maximum on the boundary (when $|x| = |y| = 1$), so that in the interior we have $h(x, y) < \frac{1}{2}$. Since $v(x, y) \leq 0$ on U , this implies $u(x, y) < \frac{1}{2}$ on U as well.

[1.2.14]

- (a) Suppose that u_1 and u_2 are two differentiable solutions with compact support: $\Delta u_1 = \Delta u_2 = f$. Then $u := u_1 - u_2$ satisfies $\Delta u = 0$. Of course, u also has compact support, and thus there is some $R > 0$ such that $u \equiv 0$ outside the closed ball B_R around the origin. Since u is continuous, it is bounded on B_R , and hence by Liouville's theorem u is constant. Since it is zero outside B_R , we see $u \equiv 0$, and thus $u_1 \equiv u_2$.

- (b) The Newtonian potential is the fundamental solution to the Laplace equation: $\Delta N = \delta_0$, the Dirac delta distribution. Because u is a smooth function with compact support, it is a valid test function. We can therefore evaluate convolution in the sense of distributions. Since $\Delta N = \delta_0$ (the Dirac delta distribution), applying the distributional derivative transfer rule gives us

$$u = u * \delta_0 = u * \Delta N = \Delta u * N = f * N$$

[1.2.15] Let $r = \int_0^1 f(x)dx$ and define

$$g(x) = \int_0^x (f(t) - r)dt$$

Then $g(0) = 0$ by definition, and $g(1) = 0$ from the definition of r . Since f is periodic, so is g , and g is smooth because f is. We have $g' = f - r$, and thus the 1-form $(f - r)dx$ is exact.

[1.2.16]

- (a) Note that we can write

$$\begin{aligned} y_1(x) &= a + \int_0^x F(s, y_1(s))ds \\ y_2(x) &= b + \int_0^x F(s, y_2(s))ds \end{aligned}$$

Thus, using the Lipschitz condition we see

$$\begin{aligned} |y_1(x) - y_2(x)| &= \left| a - b + \int_0^x (F(s, y_1(s)) - F(s, y_2(s))) ds \right| \\ &\leq |a - b| + \left| \int_0^x (F(s, y_1(s)) - F(s, y_2(s))) ds \right| \\ &\leq |a - b| + \left| \int_0^x |F(s, y_1(s)) - F(s, y_2(s))| ds \right| \\ &\leq |a - b| + M \left| \int_0^x |y_1(s) - y_2(s)| ds \right| \end{aligned}$$

- (b) Let K be the maximum of $|y_1(x) - y_2(x)|$ on the interval $\left[\frac{-1}{2M}, \frac{1}{2M}\right]$, and note that there is a point x_0 in that interval where this maximum is achieved. From the inequality in the previous part, we have

$$K \leq |a - b| + M \left| \int_0^{x_0} |y_1(s) - y_2(s)| ds \right| \leq MK |x_0|$$

Since $x_0 \in \left[-\frac{1}{2M}, \frac{1}{2M}\right]$, the above shows $K \leq |a - b| + \frac{K}{2}$, which rearranges to the inequality we want.

1.3 Fourier Series

[1.3.1] Since $f(x)$ is smooth, we can write the Fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

Now the hypothesis of the problem implies $c_0 = 0$; since $f(x)$ is not identically zero, we know there exists $k \neq 0$ with $c_k \neq 0$. Now the solution to the heat equation $u_t = u_{xx}$ with the Fourier series for f gives

$$u(x, t) = \sum_{n=-\infty}^{\infty} c_n e^{-n^2 t} e^{inx}$$

To see this solution works, note that $t > 0$ implies $e^{-n^2 t}$ decays exponentially, so we can differentiate the above sum term by term to verify $u_t = u_{xx}$. Since $f \in L^2(S^1)$, we apply full periodic boundary conditions, allowing us to expand $u(x, t)$ in a Fourier series. Then Parseval's identity shows we have

$$\begin{aligned} \|f\|_2^2 &= 2\pi \sum_{n \neq 0} |c_n|^2 \\ \|u(\cdot, t)\|_2^2 &= 2\pi \sum_{n \neq 0} |c_n|^2 e^{-2n^2 t} \end{aligned}$$

Because $t > 0$ and $n \neq 0$, we know $e^{-2n^2 t} < 1$. This shows $\|u(\cdot, t)\|_2 \leq \|f\|_2$. We get strict inequality noting that $c_k \neq 0$, so the $n = k$ term of the sum is strictly smaller for $u(\cdot, t)$ than for f .

[1.3.2] Suppose such an f existed, and write $\hat{g}(k)$ for the k th Fourier coefficient of g . Then we have

$$\hat{f}(k)\hat{g}(k) = \widehat{f * g}(k) = \hat{g}(k)$$

Here the first equality follows because we're convolving on S^1 , and the second follows from the assumed identity $f * g = g$. Now if $g_n(x) = e^{2\pi i n x}$, then $g_n \in L^1(S^1)$ and $\hat{g}_n(k)$ is 1 when $k = n$, and 0 otherwise. By the above then, we have $\hat{f}(k) = 1$ for all $k \in \mathbb{Z}$. This contradicts the Riemann-Lebesgue lemma: since $f \in L^1(S^1)$, we should have $\lim_{|k| \rightarrow \infty} \hat{f}(k) = 0$.

1.3 Fourier Series

[1.3.3] Let $f(x) = \sum c_n e^{2\pi i n x}$ be the Fourier expansion for f . By hypothesis we know $c_0 = 0$. Because f is C^1 , f' is continuous and therefore in $L^2(S^1)$. We can find its Fourier coefficients via integration by parts:

$$\begin{aligned} c'_n &= \int_0^1 f'(x) e^{-2\pi i n x} dx \\ &= \left[f(x) e^{-2\pi i n x} \right]_0^1 - \int_0^1 f(x) (-2\pi i n) e^{-2\pi i n x} dx \\ &= 0 + 2\pi i n \int_0^1 f(x) e^{-2\pi i n x} dx \\ &= 2\pi i n c_n \end{aligned}$$

Now, apply Parseval's identity to f' :

$$\|f'\|_2^2 = \sum_{n \neq 0} |c'_n|^2 = \sum_{n \neq 0} |2\pi i n c_n|^2 = 4\pi^2 \sum_{n \neq 0} n^2 |c_n|^2$$

Because n is a non-zero integer, $n^2 \geq 1$ for all terms in the sum. Therefore:

$$\|f'\|_2^2 \geq 4\pi^2 \sum_{n \neq 0} |c_n|^2 = 4\pi^2 \|f\|_2^2$$

Taking square roots gives our desired inequality.

[1.3.4]

(a) Let $F(z) = (1 - z)^{-1/4}$ be the principal branch of the power function, defined on the open unit disk U . Since F is analytic on U , we can write its Taylor series expansion using the binomial theorem:

$$\begin{aligned} (1 - z)^{-1/4} &= \sum_{n=0}^{\infty} \binom{-1/4}{n} (-z)^n \\ &= 1 + \sum_{n=1}^{\infty} \frac{1 \cdot 5 \cdot 9 \cdots (4n-3)}{4^n n!} z^n \end{aligned}$$

Since $r < 1$, re^{it} is in U , meaning $f_r(t) = F(re^{it})$. So we can write

$$f_r(t) = 1 + \sum_{n=1}^{\infty} \frac{1 \cdot 5 \cdot 9 \cdots (4n-3)}{4^n n!} r^n e^{int}$$

Because $r < 1$, the Weierstrass M-test shows this series converges uniformly and absolutely to $f_r(t)$. This is then the Fourier series for $f_r(t)$.

1.3 Fourier Series

- (b) We first express $|1 - e^{it}|$ in a form more amenable to taking powers. Using trig identities we have

$$\begin{aligned} |1 - e^{it}|^2 &= (1 - \cos t)^2 + (-\sin t)^2 \\ &= 1 - 2\cos t + \cos^2 t + \sin^2 t \\ &= 2 - 2\cos t \\ &= 4\sin^2(t/2) \end{aligned}$$

Thus $|1 - e^{it}| = 2\sin(t/2)$. Now for $t \in [0, \pi]$, we have $\pi \sin(t/2) \geq t$, and similarly for $t \in [\pi, 2\pi]$, we have $\pi \sin(t/2) \geq 2\pi - t$. This yields

$$\begin{aligned} \int_0^{2\pi} |f_1(t)| dt &= \int_0^\pi |1 - e^{it}|^{-1/4} dt + \int_\pi^{2\pi} |1 - e^{it}|^{-1/4} dt \\ &= \int_0^\pi (2\sin(t/2))^{-1/4} dt + \int_\pi^{2\pi} (2\sin(t/2))^{-1/4} dt \\ &\leq C_1 \int_0^\pi t^{-1/4} dt + C_2 \int_\pi^{2\pi} (2\pi - t)^{-1/4} dt \end{aligned}$$

Here C_1 and C_2 are constants we don't really care about. Both of these final integrals can be evaluated to finite values, and thus $\|f_1\|_1 < \infty$, meaning that $f_1 \in L^1([0, 2\pi])$.

- (c) We have that $f_r \rightarrow f_1$ as $r \rightarrow 1^-$ whenever $t \in (0, 2\pi)$ by continuity of the principal branch of the power function away from the branch cut. Thus $f_r \rightarrow f_1$ almost everywhere in $[0, 2\pi]$. This is the first ingredient necessary for applying the Dominated Convergence Theorem.

Using the same trig identities as in the previous part, we have

$$|1 - re^{it}|^2 = (1 - r)^2 + 4r \sin^2(t/2) \geq 4r \sin^2(t/2)$$

This shows that for $r \geq \frac{1}{4}$, we have $|f_r(t)| \leq |g(t)|$, where $g(t) = \sin(t/2)^{-1/4}$. The proof of the previous part showed that $g \in L^1([0, 2\pi])$. Thus $g(t)$ is an integrable function that dominates $|f_r(t)|$ (and hence $|f_r(t)e^{-int}|$) for $r \geq \frac{1}{4}$. This is the second (and final) ingredient necessary for applying the Dominated Convergence theorem, and we have

$$\lim_{r \rightarrow 1^-} \frac{1}{2\pi} \int_0^{2\pi} f_r(t) e^{-int} dt = \frac{1}{2\pi} \int_0^{2\pi} \lim_{r \rightarrow 1^-} (f_r(t) e^{-int}) dt = \frac{1}{2\pi} \int_0^{2\pi} f_1(t) e^{-int} dt$$

Thus our Fourier coefficients are the limits of the ones in part (a) as $r \rightarrow 1^-$. Explicitly, all negative coefficients are 0, $c_0 = 1$, and for $n > 0$ we have

$$c_n = \frac{1 \cdot 5 \cdot 9 \cdots (4n - 3)}{4^n n!}$$

1.3 Fourier Series

(a) We use Cauchy-Schwarz to see

$$\left(\sum_{n \neq 0} |\hat{f}(n)| \right)^2 = \left(\sum_{n \neq 0} |n|^{-s} |n|^s |\hat{f}(n)| \right)^2 \leq \sum_{n \neq 0} |n|^{-2s} \sum_{n \neq 0} |n|^{2s} |\hat{f}(n)|^2$$

The sums on the right are both finite: the first by the exponent/comparison test (here we use $2s > 1$), the second by hypothesis. Thus $\sum_{n=-\infty}^{\infty} |\hat{f}(n)|$ is finite as well (here we know $\hat{f}(0)$ is finite because $f \in L^1[0, 2\pi]$).

(b) Write

$$S_N(x) = \sum_{n=-N}^N \hat{f}(n) e^{inx}$$

for the partial symmetric sums of the Fourier series for f . By part (a) (and the fact $|\hat{f}(n)| = |\hat{f}(n) e^{inx}|$), these terms are absolutely summable, and then by the Weierstrass M-test $g(x) = \lim_{N \rightarrow \infty} S_N(x)$ is a continuous function, with $S_N(x)$ converging uniformly. Because $S_N(x)$ is 2π -periodic for all N , so is $g(x)$. Let's evaluate the Fourier coefficients for g , using this uniform convergence:

$$\hat{g}(k) = (2\pi)^{-1} \int_0^{2\pi} g(x) e^{-ikx} dx = \lim_{N \rightarrow \infty} (2\pi)^{-1} \int_0^{2\pi} S_N(x) e^{-ikx} dx = \lim_{N \rightarrow \infty} \hat{f}(k) = \hat{f}(k)$$

Thus g and f have the same Fourier coefficients. Since they are both in $L^1[0, 2\pi]$ and the Fourier transform is injective on L^1 , they agree almost everywhere in $[0, 2\pi]$, and hence g and F agree almost everywhere as well.

(c) Let us write

$$h(x) = \sum_{n=-\infty}^{\infty} in \hat{f}(n) e^{inx}$$

which is the "formal" derivative of g ; we will show it is continuous and the actual derivative of g , which (by part (b)) will show F is almost everywhere equal to a continuously differentiable function.

To do this, note that $h(x) = \lim_{N \rightarrow \infty} S'_N(x)$. So if we show this is uniform convergence, then we will have that $h(x)$ is continuous, and that $g'(x) \equiv h(x)$; this follows because $S_N(x) \rightarrow g(x)$ and $S'_N(x) \rightarrow h(x)$, both uniformly. But this follows from the Weierstrass M-test, as above, again using Cauchy-Schwarz:

$$\left(\sum_{n \neq 0} |n \hat{f}(n)| \right)^2 = \left(\sum_{n \neq 0} |n|^{1-s} |n|^s |\hat{f}(n)| \right)^2 \leq \sum_{n \neq 0} |n|^{2-2s} \sum_{n \neq 0} |n|^{2s} |\hat{f}(n)|^2$$

One sum on the right is finite by hypothesis, and the other because $2s - 2 > 1$, since $s > \frac{3}{2}$.

[1.3.6] No. If $f \in C(S^1)$ has these coefficients, then $f \in L^2(S^1)$, and so by Parseval's theorem $\sum c_n^2 < \infty$. But we have

$$\sum_{n \in \mathbb{Z}} c_n^2 = \sum_{n \in \mathbb{Z}} \frac{1}{1 + 2\sqrt{|n|} + |n|} \geq \sum_{n \in \mathbb{Z}} \frac{1}{1 + 3|n|} \geq 1 + 2 \sum_{n > 0} \frac{1}{4n}$$

And the latter sum diverges as a constant multiple of the harmonic series. This contradiction shows f does not exist.

[1.3.7] First assume $f(x)$ is an odd function. Then we have

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx = 0$$

where the last equality follows because $\cos$ is even, so $f(x) \cos(nx)$ is still an odd function.

Conversely, assume all the a_n are zero. Because $f \in C^2(\mathbb{R})$, its Fourier coefficients are absolutely summable: the coefficients decay as $O(n^{-2})$ via integration by parts, and $\sum n^{-2}$ is finite. So by the Weierstrass M-test, its Fourier series converges uniformly and absolutely. Uniqueness of Fourier series shows f is equal to this sum almost everywhere. Since the sum and f are both continuous, we actually have pointwise equality everywhere:

$$f(x) = \sum_{n > 0} b_n \sin(nx)$$

Replacing x by $-x$ in the above and noting that $\sin$ is odd, we have $f(-x) = -f(x)$, so that f is odd as well.

[1.3.8] The Cauchy-Schwarz inequality gives

$$\left(\sum_{n \in \mathbb{Z}} |\hat{f}(n)| \right)^2 \leq \left(\sum_{n \in \mathbb{Z}} \frac{1}{1 + n^2} \right) \left(\sum_{n \in \mathbb{Z}} (1 + n^2) |\hat{f}(n)|^2 \right)$$

The first sum on the right-hand side converges by the p-series test, say to a real number C . So if $f \in H$, then the right-hand side is finite. The Weierstrass M-test then shows the Fourier series converges uniformly to a continuous function, which by uniqueness must equal f almost everywhere (and hence, by definition of L^2 , we can assume is f). Now the triangle inequality shows that

$$\left| \sum_{k=-n}^n \hat{f}(k) e^{2\pi i k x} \right| \leq \sum_{k=-n}^n |\hat{f}(k)|$$

1.3 Fourier Series

Uniform convergence then shows that $|f(x)| \leq \sum_{n \in \mathbb{Z}} |\hat{f}(n)|$, which is finite and independent of x . Thus from what was shown above, we have

$$\|f\|_\infty \leq \sum_{n \in \mathbb{Z}} |\hat{f}(n)| \leq \sqrt{C} \|f\|_H$$

[1.3.9] Fix $f \in V$ and $\epsilon > 0$. Let $s_n(f)$ be the truncated Fourier series for f , and $\sigma_n(f)$ the Cesàro sums:

$$s_n(f) = \sum_{k=-n}^n \hat{f}(k) e^{ikx}$$

$$\sigma_n(f) = \frac{1}{n+1} \sum_{k=0}^n s_k(f)$$

The hypothesis that $f \in V$ implies that $\hat{f}(n) = 0$ when $n \notin I$. Thus $s_n(f)$, and also $\sigma_n(f)$, are in S . By Fejér's Theorem, $\sigma_n(f)$ converges to f uniformly, and thus we can choose N large enough that $\|f - \sigma_N(f)\|_\infty < \epsilon$. Since f and ϵ were arbitrary, this shows S is dense in V .

[1.3.10] Choose $g \in C^\infty(S^1)$, and let $\hat{g}(n)$ denote the n th Fourier coefficient of g . Thus we can write

$$\hat{g}(n) = \frac{1}{2\pi} \int_0^{2\pi} g(x) e^{-inx} dt$$

Define a new sequence of Fourier coefficients as $\hat{f}(n) = \hat{g}(n)/p(in)$; the hypotheses guarantee we aren't dividing by zero here. Because $g \in C^\infty(S^1)$, its Fourier coefficients have rapid decay: for any integer $N \geq 0$, there exists a constant C_N such that $|\hat{g}(n)| \leq C_N |n|^{-N}$ for all $n \neq 0$. Because $|p(in)|$ grows at worst polynomially, $|\hat{f}(n)| = |\hat{g}(n)|/|p(in)|$ also obeys bounds of the form $O(|n|^{-M})$ for all M . Therefore, the sequence $\hat{f}(n)$ has rapid decay, guaranteeing that $f(x) = \sum \hat{f}(n) e^{inx}$ is a well-defined $C^\infty(S^1)$ function. Now note that, by differentiating the Fourier series term-by-term, we have

$$f(x) = \sum_{n \in \mathbb{Z}} \hat{f}(n) e^{int}$$

$$\widehat{f'}(n) = (in) \hat{f}(n)$$

$$\widehat{f''}(n) = (in)^2 \hat{f}(n)$$

$$\dots = \dots$$

$$\widehat{f^{(k)}}(n) = (in)^k \hat{f}(n)$$

$$\widehat{Pf}(n) = p(in) \hat{f}(n)$$

The last equality here follows from the linearity of the Fourier transform. Because $p(in)\hat{f}(n) = \hat{g}(n)$, Pf and g are smooth functions on S^1 with the same Fourier coefficients, and hence $Pf \equiv g$. Since $g \in C^\infty(S^1)$ was arbitrary, this shows P is surjective.

[1.3.11] Let $\hat{f}(\xi)$ be the Fourier transform, and note that $\hat{f}_t(\xi) = e^{2\pi i t} \hat{f}(\xi)$. We have pointwise convergence (for fixed ξ)

$$\lim_{t \rightarrow 0} |\hat{f}(\xi) - \hat{f}_t(\xi)|^2 = |\hat{f}(\xi)|^2 \cdot \lim_{t \rightarrow 0} |1 - e^{2\pi i t}|^2 = 0$$

Now $|1 - e^{2\pi i t}| \leq 2$ for all $t \in \mathbb{R}$, and thus $|\hat{f}(\xi) - \hat{f}_t(\xi)|^2$ is bounded above by $4|\hat{f}(\xi)|^2$, which is integrable because $f \in L^2(\mathbb{R})$. Plancherel's theorem and the Dominated Convergence Theorem then show

$$\lim_{t \rightarrow 0} \|f - f_t\|_2^2 = \lim_{t \rightarrow 0} \int_{\mathbb{R}} |\hat{f} - \hat{f}_t|^2 d\xi = \int_{\mathbb{R}} \lim_{t \rightarrow 0} |\hat{f} - \hat{f}_t|^2 d\xi = 0$$

where the last equality follows because we showed above $|\hat{f}(\xi) - \hat{f}_t(\xi)|^2 \rightarrow 0$ for all ξ in $\mathbb{R}$.

1.4 Functional Analysis

[1.4.1] First let $e_m(x) = e^{imx}$ with $m \neq 0$. Then $\int_0^{2\pi} e_m dx = 0$. Now $e^{im\theta} \neq 1$ because θ is an irrational angle, so we have a geometric sum:

$$S_n(e_m)(x) = \frac{1}{n} \sum_{k=0}^{n-1} e^{imx} e^{imk\theta} = \frac{e^{imx}}{n} \frac{1 - e^{imn\theta}}{1 - e^{im\theta}}$$

For a fixed m , each of these factors is bounded absolutely:

$$|S_n(e_m)(x)| = \frac{1}{n} |e^{imx}| \frac{|1 - e^{imn\theta}|}{|1 - e^{im\theta}|} \leq \frac{2}{n} |1 - e^{im\theta}|^{-1}$$

This shows $S_n(e_m) \rightarrow 0$ uniformly.

Now let $f \in C(S^1)$ be arbitrary, with $\int_0^{2\pi} f dx = 0$. By the Stone-Weierstrass theorem, choose a trigonometric polynomial $p(x)$ such that $|f(x) - p(x)| < \epsilon$ for all x . We let $\hat{p}(0)$ be the constant coefficient for this polynomial; we see that

$$|\hat{p}(0)| = \frac{1}{2\pi} \left| \int_0^{2\pi} p(x) dx \right| = \frac{1}{2\pi} \left| \int_0^{2\pi} (p(x) - f(x)) dx \right| \leq \frac{1}{2\pi} \int_0^{2\pi} |p(x) - f(x)| dx < \epsilon$$

Thus if we define $q(x) := p(x) - \hat{p}(0)$, then $\|f - q\|_\infty < 2\epsilon$. Also since $\hat{q}(0) = 0$, the previous paragraph (plus linearity) shows $S_n(q) \rightarrow 0$ uniformly, so that for some $N > 0$, $n > N$ implies $\|S_n(q)\|_\infty < \epsilon$. Since trivially we have $\|S_n(g)\|_\infty \leq \|g\|_\infty$ for all $g \in C(S^1)$, the triangle inequality and linearity of S_n shows

$$\|S_n(f)\|_\infty \leq \|S_n(f - q)\|_\infty + \|S_n(q)\|_\infty \leq \|f - q\|_\infty + \epsilon < 3\epsilon$$

Thus $S_n(f) \rightarrow 0$ uniformly.

1.5 Measure Theory

[1.5.1] Since $[0, 1]$ is compact, it is enough to apply the Arzelà-Ascoli theorem, so we need to show that the f_n are uniformly bounded, and equicontinuous. Uniform boundedness is already a hypothesis. To check equicontinuity, fix $n > 0$ and let $x, y \in [0, 1]$ with $x < y$. Absolute continuity for f_n then implies

$$f_n(y) - f_n(x) = \int_x^y f'_n(t) dt$$

Using the Cauchy-Schwarz theorem we have

$$|f_n(y) - f_n(x)| \leq \int_x^y |f'_n(t)| dt \leq \left(\int_x^y (f'_n(t))^2 dt \right)^{1/2} \left(\int_x^y 1^2 dt \right)^{1/2}$$

Now $(f'_n(t))^2$ is non-negative, so we only strengthen our inequality if we extend the integral domain to the full $[0, 1]$. Combined with our hypothesis then, we have $|f_n(y) - f_n(x)|$ is bounded above by $\sqrt{N}|y - x|^{1/2}$. Equicontinuity then follows: if $\epsilon > 0$, choose $\delta = \epsilon^2/N$ (we can ignore the $N = 0$ case since then f_n is constant for all n , so of course equicontinuous). For $|y - x| < \delta$ we have $|f_n(y) - f_n(x)| < \epsilon$. Since δ does not depend on n, x , or y , the $\{f_n\}$ are equicontinuous.

[1.5.2] We define a positive function $f \in L^1(\mu)$ and define ν such that $d\nu = f d\mu$. Since X is σ -finite, we can partition it into countably many disjoint measurable subsets $\{X_n\}$ such that $\mu(X_n)$ is finite for all n . Let $I \subset \mathbb{N}$ be the indices where $\mu(X_i) > 0$, and choose positive numbers $\{w_i\}$ such that $\sum_{i \in I} w_i = 1$. Define a function $f : X \rightarrow \mathbb{R}$ as

$$f(x) = \sum_{i \in I} \frac{w_i \chi_i(x)}{\mu(X_i)}$$

where χ_n is the characteristic function for X_n . Note that f is positive except on $\cup_{i \notin I} X_i$, which has measure zero. Finally, define $\nu(A) = \int_A f d\mu$. Using the monotone convergence theorem, we see ν is a probability measure:

$$\begin{aligned} \nu(X) &= \int_X f d\mu = \int_X \sum_{n \in I} \frac{w_n}{\mu(X_n)} d\mu = \sum_{n \in I} \int_{X_n} \frac{w_n}{\mu(X_n)} d\mu \\ &= \sum_{n \in I} \frac{w_n}{\mu(X_n)} \mu(X_n) = \sum_{n \in I} w_n = 1 \end{aligned}$$

To see mutual absolute continuity, first suppose $\mu(A) = 0$. Then of course $\nu(A) = \int_A f d\mu$ is 0 as well. Conversely, suppose $\nu(A) = 0$. Since f is positive almost everywhere, it must be that we're integrating over a set of μ -measure zero, and thus $\mu(A) = 0$.

[1.5.3] Since f is compactly supported, let it be zero outside the closed interval K , and we show f is zero almost everywhere on K . Note that the hypothesis of the problem includes $\int_K f(x) dx = 0$, so that $f \in L^1(K)$. By the linearity of the integral, we see that $\int_K f(x) p(x) dx = 0$ for all polynomials $p(x)$. By the Stone-Weierstrass theorem, $\int_K f(x) g(x) dx = 0$ for all continuous functions $g \in C(K)$. Now let $h(x)$ be the sign of $f(x)$; thus $h \in L^\infty(K)$. By Lusin's theorem we can find a sequence of continuous functions $h_k \in C(K)$ such that $h_k(x) \rightarrow h(x)$ almost everywhere. In addition, since $\|h\|_\infty = 1$, we can choose our sequence so that $\|h_n\|_\infty \leq 1$ for all n . The Dominated Convergence theorem (using $|f| \in L^1(K)$) shows

$$\int_K |f(x)| dx = \int_K f(x) h(x) dx = \int_K f(x) \lim_{n \rightarrow \infty} h_n(x) dx = \lim_{n \rightarrow \infty} \int_K f(x) h_n(x) dx = 0$$

Thus the set where $f(x) \neq 0$ is of measure zero.

[1.5.4] For $k > 0$, define

$$S_k = \bigcup_{n \geq k} E_n$$

Then we have $S = \cap S_k$. Since the $\{E_n\}$ are measurable, and the measurable subsets form a σ -algebra, we see that S is measurable. Let $\epsilon > 0$ and pick an N such that

$$\sum_{n \geq N} \mu(E_n) < \epsilon$$

By subadditivity, we have $\mu(S_N) < \epsilon$, and since $S \subset S_N$, we have $\mu(S) < \epsilon$. Since ϵ was arbitrary, we see that $\mu(S) = 0$.

[1.5.5] Let $\{q_n\}$ be an enumeration of the rational numbers in $[0, 1]$, and define

$$U_n = (q_n - 2^{-(n+2)}, q_n + 2^{-(n+2)}) \cap [0, 1]$$

Then U_n is an open subset of $[0, 1]$ with $\mu(U_n) \leq 2^{-(n+1)}$. If we define $U = \cup U_n$, then U is an open subset of $[0, 1]$, with $\mu(U) \leq \frac{1}{2}$ by subadditivity. Since $\{q_n\}$ is dense in $[0, 1]$, so is U , and thus $\partial U = \bar{U} \setminus U^\circ$ is just $[0, 1] \setminus U$, and thus $\mu(\partial U) \geq 1 - \frac{1}{2}$. But ∂U is exactly the set of discontinuity points of χ_U (proven below), so by Lebesgue's criterion χ_U is not Riemann integrable.

To see that ∂X is the set of discontinuity for χ_X , note that $\chi_X \equiv 1$ on the interior X° , while $\chi_X \equiv 0$ on the exterior $\mathbb{R}^n \setminus \bar{X}$. So none of these points are in the set of discontinuity. However, if $x \in \partial X$, then every neighborhood of x intersects both X and its complement, and so χ_X is not locally constant there.

[1.5.6] We will write f_x for f restricted to U_x . Let $V \subset \mathbb{R}$ be an open subset; we want to show that $f^{-1}(V)$ is measurable. For every $x \in \mathbb{R}$, we know by hypothesis that $f_x^{-1}(V)$ is measurable. But

$$f_x^{-1}(V) = f^{-1}(V) \cap U_x$$

Now $\{U_x\}$ covers $\mathbb{R}$, and since $\mathbb{R}$ is Lindelof, there is a countable collection $\{U_n\}$ that covers $\mathbb{R}$. But then

$$f^{-1}(V) = \bigcup_{n \geq 0} f_n^{-1}(V)$$

is a countable union of measurable sets, hence measurable.

[1.5.7] To show $d(f, g)$ is a measure, first note it is well-defined because $\mu(X)$ is finite and when $[f] = [F]$ and $[g] = [G]$, it's clear that $\rho(f - g) = \rho(F - G)$. Now:

- Clearly $\rho(f - g) = \rho(g - f)$.
- If $[f] = [g]$, then the integrand is zero outside almost everywhere, so $\rho(f - g) = 0$. If $[f] \neq [g]$, then there is a set of positive measure where $f \neq g$, and on that set the integrand is strictly positive, so $\rho(f - g) > 0$.
- Since ρ' is non-negative, we certainly have $\rho(|f - h|) \leq \rho(|f - g| + |g - h|)$. But then

$$\begin{aligned} \rho(f - h) &= \rho(|f - h|) \\ &\leq \rho(|f - g| + |g - h|) \\ &= \int_X \frac{|f - g| + |g - h|}{1 + |f - g| + |g - h|} d\mu \\ &\leq \int_X \frac{|f - g|}{1 + |f - g|} d\mu + \int_X \frac{|g - h|}{1 + |g - h|} d\mu \\ &= \rho(f - g) + \rho(g - h) \end{aligned}$$

So $d(f, g)$ is a metric. Suppose $\delta, \epsilon > 0$, and note that

$$\mu(\{x \in X : |f(x) - f_n(x)| \geq \delta\}) > \epsilon \implies \int_X \frac{|f - f_n|}{1 + |f - f_n|} d\mu > \frac{\delta\epsilon}{1 + \delta}$$

Thus convergence in the metric implies convergence in measure. Conversely, if $f_n \rightarrow f$ in measure, then we can find N such that for $n \geq N$ we have

$$\mu(\{x \in X : |f(x) - f_n(x)| \geq \delta\}) < \epsilon$$

Since the integrand is bounded by 1, we then have $d(f, f_n) \leq \delta\mu(X) + \epsilon$, so that $f_n \rightarrow f$ in the metric as well.

[1.5.8] Suppose this was not true: there is an open subset $U \subset D$ such that $T^n(U) \cap U = \emptyset$ for all $n > 0$. If $k < m$, then with $n = m - k$, applying T^k to everything shows $T^m(U) \cap T^k(U) = \emptyset$. Since $T(D) = D$, all of these are contained in D . But T is volume-preserving, so that $\mu(T^n(U)) = \mu(U)$, which is positive because U is open. Then $\{T^n(U)\}$ is a countable collection of disjoint sets, all of positive measure, all contained in D . This implies $\mu(D)$ is infinite, which contradicts D being bounded. This contradiction shows the conclusion of the problem is true.

[1.5.9] We will work in $E := [0, 1] \setminus \mathbb{Q}$, to avoid ambiguous decimal expansions; this won't change any measure results, since $\mu(\mathbb{Q}) = 0$ and μ is a complete measure. For $m, n > 0$ with $m \neq n$, let $E_{m,n}$ be the set of $x \in E$ such that the first 2 in its decimal expansion is in position m , and the first 3 is in position n . As a finite union of intervals, each $E_{m,n}$ is measurable, and they are all disjoint. Then

$$E_k = \bigcup_{n>k} E_{k,n}$$

is also measurable, with

$$\mu(E_k) = \sum_{n=k+1}^{\infty} \mu(E_{k,n}) = \sum_{n=k+1}^{\infty} \left(\frac{1}{10}\right)^2 \left(\frac{8}{10}\right)^{k-1} \left(\frac{9}{10}\right)^{n-k-1} = \frac{8^{k-1}}{10^{k+1}} \sum_{j=0}^{\infty} \left(\frac{9}{10}\right)^j = \frac{8^{k-1}}{10^k}$$

So $\hat{S} = S \setminus \mathbb{Q} = \cup E_k$ is measurable, implying $S = \hat{S} \cup (S \cap \mathbb{Q})$ is measurable, with

$$\mu(S) = \mu(\hat{S}) = \sum_{j>0} \mu(E_j) = \frac{1}{10} \sum_{j>0} \left(\frac{8}{10}\right)^{j-1} = \frac{1}{2}$$

1.6 TBM

[1.6.1]

(a) For $1 \leq i \leq k$, define

$$B_i(z) = \frac{z - a_i}{1 - \bar{a}_i z}$$

Thus B is the product of the B_i . Since norm is multiplicative, it suffices to show $|B_1(z)|^2 = 1$ when $|z| = 1$. Noting that $\bar{z} = z^{-1}$ in this case, we have

$$|B_1(z)|^2 = \frac{z - a_1}{1 - \bar{a}_1 z} \frac{\bar{z} - \bar{a}_1}{1 - a_1 \bar{z}} = \frac{1 - a_1 \bar{z} - \bar{a}_1 z + |a_1|^2}{1 - a_1 \bar{z} - \bar{a}_1 z + |a_1|^2} = 1$$

(b) Since $a_i \in \mathbb{D}$, we have $|\bar{a}_i| < 1$, and thus $1 - \bar{a}_i z \neq 0$ as long as $|a_i| \cdot |z| < 1$. Thus B is holomorphic in an open neighborhood of $\overline{\mathbb{D}}$, and in particular is harmonic there. Now let $g_n(z) = z^n$ for $n \in \mathbb{Z}$. Clearly $g_0(B(z)) = g_0(z) = 1$, so their integrals match. For $n > 0$ we have that $B(z)^n$ is holomorphic, hence harmonic, in a neighborhood of $\overline{\mathbb{D}}$, and thus

$$\int_0^{2\pi} g_n(e^{it}) dt = \int_0^{2\pi} e^{int} dt = 0 = 2\pi B(0)^n = \int_0^{2\pi} g_n(B(e^{it})) dt$$

By the mean value theorem. For $n < 0$ we have

$$\int_0^{2\pi} B(e^{it})^n dt = \int_0^{2\pi} \overline{B(e^{it})^{-n}} dt = \overline{\int_0^{2\pi} B(e^{it})^{-n} dt} = \bar{0} = 0 = \int_0^{2\pi} g_n(e^{it}) dt$$

So the conclusion is true for all $\{g_n\}$. The Stone-Weierstrass theorem shows the linear span of $\{g_n\}$ is dense in $C(S^1)$, and thus for $f \in C(S^1)$ we can find a sequence $\{p_n\}$ of trigonometric polynomials converging uniformly to f . But then

$$\begin{aligned} \int_0^{2\pi} f(e^{it}) dt &= \int_0^{2\pi} \lim_{n \rightarrow \infty} p_n(e^{it}) dt = \lim_{n \rightarrow \infty} \int_0^{2\pi} p_n(e^{it}) dt = \lim_{n \rightarrow \infty} \int_0^{2\pi} p_n(B(e^{it})) dt \\ &= \int_0^{2\pi} \lim_{n \rightarrow \infty} p_n(B(e^{it})) dt = \int_0^{2\pi} f(B(e^{it})) dt \end{aligned}$$