

The point of this short note is to prove the following

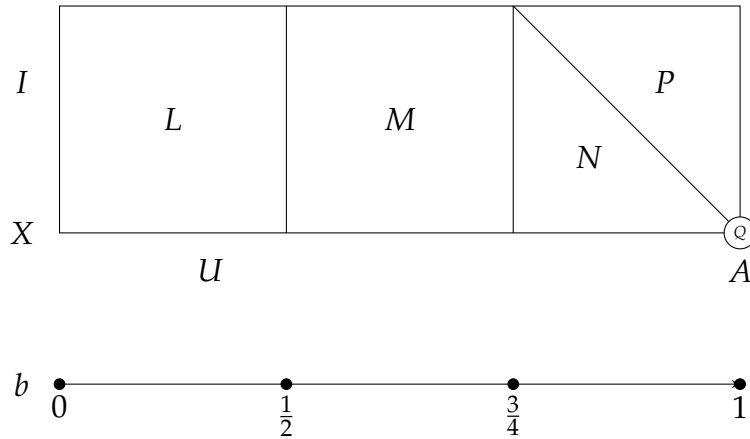
THEOREM 1: *Let $A \subset X$ be a closed subset of the topological space X , and assume there's an open neighborhood U containing A such that:*

1. *There exists a bump function $b : X \rightarrow [0, 1]$ for A in U .*
2. *There exists a deformation retraction $D : U \times [0, 1] \rightarrow X$ of U to A in X .*

Then the pair (X, A) satisfies the homotopy extension property.

The bump function here means that $b^{-1}(1) = A$ and $X \setminus U \subset b^{-1}(0)$. The deformation retraction is such that $D(u, 0) = u$, $D(u, 1) \in A$, and $D(a, t) = a$ for all $a \in A$ and $t \in [0, 1]$. Also recall that a pair (X, A) satisfies the homotopy extension property precisely when there is a retraction $r : X \times [0, 1] \rightarrow (X \times \{0\}) \cup (A \times [0, 1])$. We will write R for the image of this retraction.

Proof. We define the retraction $r : X \times [0, 1] \rightarrow R$ piecemeal. The idea of the proof can be summarized in the following picture (writing $I = [0, 1]$ throughout this proof):



Here the horizontal axis is X , and the vertical axis is I . For $x \in X$ we note that the vertical slice $\{x\} \times I$ will be mapped to a path in R . We will describe the path within each labeled section of $X \times I$ below.

We define $L = \{(x, t) \mid b(x) \leq \frac{1}{2}\}$. These are points of X we can consider "far" from A . These points have vertical slices that get mapped to constant paths via projection:

$$r_L(x, t) = (x, 0)$$

This is clearly continuous, and is the identity on $L \cap R$, since $L \cap (A \times I) = \emptyset$.

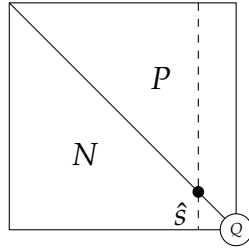
We define $M = \{(x, t) \mid \frac{1}{2} \leq b(x) \leq \frac{3}{4}\}$. The retraction on M is

$$r_M(x, t) = (D(x, 2\hat{b}(x)t), 0)$$

Note this makes sense because when $b(x) > 0$, then $x \in U$. Here $\hat{b}(x) = \max(2b(x) - 1, 0)$ is a dampened version of b which ensures r is continuous on $L \cap M$. It means that points in U that are still "far" from A don't follow their D -path at all. The elements in M are then travelling partially along their D -path, while staying in $X \times \{0\}$. This is continuous, and once again is the identity on $M \cap R$, since $D(x, 0) = x$ for all $x \in U$. If $(x, t) \in L \cap M$, then $\hat{b}(x) = 0$, and so

$$r_M(x, t) = (D(x, 0), 0) = (x, 0) = r_L(x, t)$$

Once our points get to the $b(x) \geq \frac{3}{4}$ area, they are fairly "close" to A . Now instead of just traversing their D -path, we want to traverse the path and then vertically climb in $A \times I$. To make this work, we need to traverse the D -path at an increased speed, and then climb at a unit speed.



In the diagram above, $\hat{s} = 2(1 - \hat{b})$ is such that $\hat{s}^{-1}$ is the "speed" we traverse the D -path for that vertical slice. As we climb the vertical slice, we do the full D -path on the segment contained in N , and then climb up $D(x, 1) \times I$ at unit speed on the segment contained in P . To be precise:

$$\begin{aligned} N &= \{(x, t) \mid t \leq \hat{s}(x) \leq 1\} & P &= \{(x, t) \mid t \geq \hat{s}(x)\} \\ r_N(x, t) &= (D(x, t\hat{s}^{-1}(x)), 0) & r_P(x, t) &= (D(x, 1), t - \hat{s}(x)) \end{aligned}$$

There's a problem here though: when $\hat{s}(x) = 0$ (which happens precisely when $x \in A$), then r_N is not defined. We thus split off this problematic piece as $Q = A \times \{0\}$, and redefine N as $N \setminus Q$. We'll show continuity on Q below. Note that on $M \cap N$ we have $2\hat{b}(x) = \hat{s}(x) = 1$, and thus

$$r_M(x, t) = (D(x, t), 0) = r_N(x, t)$$

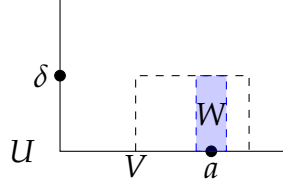
Also, $N \cap P$ is when $\hat{s}(x) = t$, and then we have

$$r_N(x, t) = (D(x, 1), 0) = r_P(x, t)$$

We also see that $N \cap R$ is contained in $X \times \{0\}$ (remember we removed Q from N). Meanwhile we have $P \cap R = A \times I$. We check the maps above are the identity here:

$$\begin{aligned} r_N(x, 0) &= (D(x, 0), 0) = (x, 0) \\ r_P(a, t) &= (D(a, 1), t) = (a, t) \end{aligned}$$

Now since $Q \subset P$, we see that $L \cup M \cup N \cup P = X \times I$. So r so defined is a retraction to R , and it is continuous except perhaps on Q . We show now that it is continuous there too. The key ideas are that for $a \in A$, the D -path is constant, and we also use the fact that I is compact.



Let V be an open subset of U containing $a \in A$, and let δ be such that $0 < \delta < \frac{1}{4}$. Now $D(\{a\} \times I) = \{a\} \subset V$, and thus by the tube lemma there exists an open $W \subset V$ such that $D(W \times I) \subset V$. We can further shrink W so that for every $x \in W$, we have $b(x) > 1 - \delta$. This ensures $W \times I$ is a subset of $N \cup P$. Now clearly $(a, 0) \in W \times [0, \delta)$, so let $(x, t) \in W \times [0, \delta)$ be arbitrary. We have

$$\begin{aligned} r_N(x, t) &= (D(x, t\hat{s}^{-1}(x)), 0) \in V \times \{0\} \\ r_P(x, t) &= (D(x, 1), t - \hat{s}(x)) \in V \times [0, \delta) \end{aligned}$$

Thus we have that $r(W \times [0, \delta)) \subset V \times [0, \delta)$. Since V and δ were arbitrary, r is continuous at $(a, 0)$. Since $(a, 0)$ was an arbitrary point of Q , r is continuous on Q , and the proof is complete. $\square$