

Introduction

Let M be a closed orientable 3-manifold. Then it is well-known that M is the boundary of a 4-manifold. Obviously we can limit attention to the case M is connected. The usual way of proving this is to show that M can be built from S^3 via a finite sequence of Dehn surgeries. If we take the "movies" of these surgeries and stack them together, we get a cobordism between M and S^3 . Filling the S^3 with a ball gives a 4-manifold with boundary M .

This note aims to give a different proof of the result, relying on realizability of cohomology classes. The strategy is to embed M into a higher-dimensional manifold, and then build two representatives of the same cohomology class. These end up being cobordant, and that allows us to realize M as a boundary of a 4-manifold. All (co)homology below is over $\mathbb{Z}$.

Trivial Bundles

By the Whitney Embedding Theorem, we know that M embeds in $\mathbb{R}^6$, so it also embeds in the one-point compactification S^6 . We write NM for the normal bundle of $M \subset S^6$, and we think of NM as a tubular neighborhood of M in S^6 . In particular, we let $SM = \partial(NM)$ be the S^2 bundle over M . In this section we will prove that NM (and hence SM) are trivial bundles, so that $NM \cong M \times \mathbb{R}^3$ and $SM \cong M \times S^2$.

LEMMA 1: *TM is a trivial bundle.*

Proof. Since M is orientable, we know that the first Steifel-Whitney class $w_1(TM) = 0$. Because M is 3-dimensional, Wu's formula shows that $w_2(TM) = 0$ as well. Now TM is an orientable bundle of rank 3 over M , so it has structure group $SO(3)$. We can then take a local trivialization and try to extend it over the skeletons of M :

- The obstruction to extending it over the 1-skeleton is $w_1(TM) \in H^1(M, \pi_0(SO(3)))$, which we've established is zero.
- The obstruction to extending it over the 2-skeleton is $w_2(TM) \in H^2(M, \pi_1(SO(3)))$, which we've also established is zero.
- The obstruction to extending it to the 3-skeleton (and hence all of M) is given by a class in $H^3(M, \pi_2(SO(3)))$. But $SO(3) \cong \mathbb{RP}^3$ is universally covered by S^3 , so $\pi_2(SO(3)) \cong \pi_2(S^3) = 0$, and hence this obstruction is also zero.

Thus we have no obstruction to realizing a global trivialization of TM , so it is a trivial bundle. $\square$

Now TS^6 is stably trivial. This is because $S^6 \subset \mathbb{R}^7$ shows that $TS^6 \oplus NS^6 \cong T\mathbb{R}^7|_{S^6}$. But the tangent bundle of $\mathbb{R}^7$ is of course trivial, and NS^6 is a line bundle over the orientable S^6 , so it also trivial. We can thus deduce

LEMMA 2: NM is a trivial bundle.

Proof. From $TM \oplus NM \cong TS^6|_M$ and the above, we see that NM is stably trivial, so that all of its Steifel-Whitney classes are zero. Just as in the proof of the previous lemma, we can extend a local trivialization of NM to a global trivialization over the skeletons of M . Since $w_1(NM) = w_2(NM) = 0$ and $\pi_2(SO(3)) = 0$, the same reasoning as above shows NM has a global trivialization. $\square$

As mentioned in the introduction, we will be making use of cohomology classes to prove that M bounds a 4-manifold. We wcan use the above results to connect our main cohomology groups. We define $C := S^6 \setminus NM$ and note that $\partial C = SM$. We will also use i_x to denote the inclusion into the space X , and π_X to denote projection (from a product) onto a space X .

PROPOSITION 3: There is an isomorphism $\Gamma : H^2(M) \oplus H^2(C) \rightarrow H^2(SM)$, given by

$$\Gamma(\alpha, \beta) = i_C^* \beta - \pi_M^* \alpha$$

Also there is an isomorphism $\Psi : H^2(SM) \rightarrow H^2(M) \oplus H^2(S^2)$, with $\Psi\Gamma(\alpha, 0) = (-\alpha, 0)$.

Proof. First note that $S^6 = C \cup NM$, so by Mayer-Vietoris we have

$$\cdots \longrightarrow H^2(S^6) \longrightarrow H^2(C) \oplus H^2(NM) \longrightarrow H^2(SM) \longrightarrow H^3(S^6) \longrightarrow \cdots$$

Of course $H^2(S^6)$ and $H^3(S^6)$ are both trivial. Since NM is trivial, π_M^* is an isomorphism from $H^2(M)$ to $H^2(NM)$. We thus have a sequence of isomorphisms

$$H^2(M) \oplus H^2(C) \xrightarrow{(\pi_M^*, 1)} H^2(NM) \oplus H^2(C) \xrightarrow{-i_{NM}^* + i_C^*} H^2(SM)$$

The composition of these is Γ .

Now all (co)homology groups of S^2 are free Abelian, and $H^1(S^2) = 0$. The Künneth formula shows

$$H^2(SM) \cong H^2(M \times S^2) \cong H^2(M) \oplus H^2(S^2)$$

If Ψ is the isomorphism, then note that by definition $\Psi^{-1}(\alpha, 0) = \pi_M^* \alpha$, which is exactly $-\Gamma(\alpha, 0)$. $\square$

Realizing Cohomology Classes

We will now start constructing submanifolds that realize some cohomology classes, using the isomorphisms Γ and Ψ from the previous proposition. Let's first remark what *realizability* means here; we focus only on the case of H^2 , although much of what we say is based on far more general work by Thom.

THEOREM 4: *Let X be a compact orientable smooth manifold of dimension m , and $\gamma \in H^2(X)$. Then the Poincaré-Lefschetz dual in $H_{m-2}(X, \partial X)$ can actually be realized by a codimension-2 submanifold $Y \subset X$ (i.e., $\langle \gamma, [Y] \rangle_X = 1$). Moreover, if $Z \subset X$ is another codimension-2 co-oriented submanifold such that $[Z]$ is dual to γ , then Y and Z are cobordant.*

Proof. First we focus on realizing the dual of γ . Recall that $\mathbb{CP}^\infty$ is a $K(\mathbb{Z}, 2)$ space, so that γ is represented by the homotopy class of some continuous map $f : X \rightarrow \mathbb{CP}^\infty$. By cellular approximation and smoothing, we can suppose $f : X \rightarrow \mathbb{CP}^m$ is smooth. We also further homotope f and $f|_{\partial X}$ so that they are transverse to $\mathbb{CP}^{m-1}$. Since $\mathbb{CP}^{m-1}$ is Poincaré dual to the fundamental class $\mu \in H^2(\mathbb{CP}^m)$, we have that $Y := f^{-1}(\mathbb{CP}^{m-1})$ is Poincaré dual to $f^*\mu = \gamma$.

Now let $Z \subset X$ be another codimension-2 co-oriented submanifold such that $[Z]$ is dual to γ . Note we can construct Z in the same way as Y , as follows. Let $\xi \rightarrow \mathbb{CP}^\infty$ be the universal $SO(2)$ bundle, because $\mathbb{CP}^\infty$ is **also** a $BSO(2)$. Since Z is codimension-2 and co-oriented in X , the normal bundle NZ is an $SO(2)$ bundle, so is $g^*\xi$ for some smooth map $g : Z \rightarrow \mathbb{CP}^\infty$. Again we use cellular approximation and smoothing to retarget as $g : X \rightarrow \mathbb{CP}^{m-1}$. As above we consider $NZ \subset X$ as a tubular neighborhood. We can then form a map $\hat{g} : X \rightarrow \hat{\xi}$, where $\hat{\xi}$ is the one-point compactification of ξ , and $\hat{g}$ is the natural map $g^*\xi \rightarrow \xi$ on NZ , and sends everything else to the point at infinity in $\hat{\xi}$. Since $\hat{\xi} \cong \mathbb{CP}^m$, we have a smooth map $\hat{g} : X \rightarrow \mathbb{CP}^m$ such that $Z = \hat{g}^{-1}(\mathbb{CP}^{m-1})$. Now since $[Y]$ and $[Z]$ are both dual to γ , they are homotopic, implying there exists a homotopy $H : X \times [0, 1] \rightarrow \mathbb{CP}^m$ between f and $\hat{g}$. Making H transverse to $\mathbb{CP}^{m-1}$ without changing what it does on the boundary, we see that $H^{-1}(\mathbb{CP}^{m-1})$ is a cobordism between Y and Z . $\square$

Now consider the isomorphism $\Psi\Gamma : H^2(M) \oplus H^2(C) \rightarrow H^2(M) \oplus H^2(S^2)$. As proved above, $\Psi\Gamma$ is the negation mapping on $H^2(M)$. Now pick an $(\alpha, \beta) \in H^2(M) \oplus H^2(C)$ mapping to the fundamental class $\mu \in H^2(S^2)$. Then we have $\beta \mapsto (\alpha, \mu)$. The key point is that β maps to something with μ in the second coordinate. From the above theorem, we can realize α via a smooth map $F : M \rightarrow \mathbb{CP}^3$. In fact, cellular approximation and smoothing shows we can retarget F to land in the 3-skeleton of $\mathbb{CP}^3$, which is just S^2 . We define $G : M \rightarrow SM \cong M \times S^2$ as

$$G(x) = (x, F(x))$$

Note that $G(M)$ is the graph of F , and that π_M is a diffeomorphism between $G(M)$ and M .

LEMMA 5: If $\gamma \in H^2(SM)$ is dual to $[G(M)]$, then $\Psi(\gamma) = (\alpha, \mu)$.

Proof. We will show that γ and $\pi_M^* \alpha + \pi_{S^2}^* \mu$ evaluate the same on all elements of $H_3(SM)$; the non-degeneracy of the dual pairing then shows

$$\Psi(\gamma) = \Psi(\pi_M^* \alpha + \pi_{S^2}^* \mu) = (\alpha, \mu)$$

Since Ψ is an isomorphism, we can write $\gamma = \pi_M^* \delta + k \pi_{S^2}^* \mu$ for $\delta \in H^2(M)$ and $k \in \mathbb{Z}$; our ultimate goal then is to show $k = 1$ and $\delta = \alpha$. Let ν be a fundamental class for $H^3(M)$. Then we have

$$\begin{aligned} \langle \gamma \smile \pi_M^* \nu, [SM] \rangle &= \langle (\pi_M^* \delta + k \pi_{S^2}^* \mu) \smile \pi_M^* \nu, [SM] \rangle \\ &= \langle \pi_M^* (\delta \smile \nu), [SM] \rangle + (-1)^{3 \cdot 2} k \langle \pi_M^* \nu \smile \pi_{S^2}^* \mu, [SM] \rangle \\ &= k \end{aligned}$$

where the last equality follows from $\delta \smile \nu \in H^5(M) = 0$, $SM \cong M \times S^2$, and ν and μ being fundamental classes. But γ is dual to $[G(M)]$, so we also have (recalling $\pi_M G$ is the identity map)

$$k = \langle \pi_M^* \nu, [G(M)] \rangle = \langle \nu, M \rangle = 1$$

Now let $\omega \in H^1(M)$ be any element, and we compute

$$\begin{aligned} \langle \gamma \smile (\pi_M^* \omega \smile \pi_{S^2}^* \mu), [SM] \rangle &= \langle (\pi_M^* \delta + k \pi_{S^2}^* \mu) \smile \pi_M^* \omega \smile \pi_{S^2}^* \mu, [SM] \rangle \\ &= \langle \pi_M^* (\delta \smile \omega) \smile \pi_{S^2}^* \mu, [SM] \rangle \\ &= \langle \delta \smile \omega, [M] \rangle \end{aligned}$$

where we've used in the second equality that $\mu \smile \mu \in H^4(S^2) = 0$, and in the third that $SM \cong M \times S^2$ with μ a fundamental class. Again, using that γ is dual to $[G(M)]$, we have

$$\langle \delta \smile \omega, [M] \rangle = \langle \pi_M^* \omega \smile \pi_{S^2}^* \mu, [G(M)] \rangle = \langle \omega \smile F^* \mu, M \rangle = \langle \omega \smile \alpha, M \rangle$$

Here we've used that $\pi_M G$ is the identity map, whereas $\pi_{S^2} G \equiv F$. Since ω was arbitrary, and both the cup product and $\langle \rangle_M$ are non-degenerate, we have $\delta = \alpha$. $\square$

Using Theorem 4, let $P \subset C$ be a 4-manifold that is Poincaré dual to $\beta \in H^2(C)$. Since Poincaré-Lefschetz duality is natural with respect to the boundary maps, and recalling that $\partial C = SM$, there is a commutative diagram

$$\begin{array}{ccc} H^2(C) & \xrightarrow{i_C^*} & H^2(SM) \\ PD \downarrow & & \downarrow PD \\ H_4(C, SM) & \xrightarrow{\partial_*} & H_3(SM) \end{array}$$

Recall β was chosen so that $\Psi_{\mathcal{C}}^*(\beta) = (\alpha, \mu)$. Writing D for the duality maps, commutativity of the diagram implies

$$D\Psi^{-1}(\alpha, \mu) = Di_{\mathcal{C}}^*(\beta) = \partial_* D(\beta) = \partial_*([P]) = [\partial P]$$

Thus $\partial P \subset SM$ is dual to $\Psi^{-1}(\alpha, \mu)$, just as $G(M)$ is. Again by [Theorem 4](#), $G(M)$ and ∂P are cobordant. But we can fill in ∂P with P , leaving a 4-manifold bounded by $G(M) \cong M$, and we've shown

THEOREM 6: *If M is a closed orientable 3-manifold, then it bounds a 4-manifold.*